MECÁNICA CLÁSICA I

Definiciones previas

- **Posición**: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Lugar que ocupa la partícula en el espacio en un momento concreto

$$\boxed{\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}}$$

- **Trayectoria** (posición en función del tiempo)

- **Velocidad**: $\boxed{\vec{v} = \frac{d\vec{r}}{dt}}$ Variación de la posición en función del tiempo

$\vec{v}(t)$ siempre tangente a $\vec{r}(t)$

$\boxed{\vec{v} = v\vec{\tau}}$ → vector unitario tangente a la trayectoria

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt}$$

- **Aceleración**: $\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}}$ Variación de la velocidad en función del tiempo

$$\boxed{\vec{a} = \frac{d(v\vec{\tau})}{dt} = \frac{dv}{dt}\vec{\tau} + v\frac{d\vec{\tau}}{dt}}$$

→ **aceleración tangencial**: relacionada con la variación de la celeridad ($|\vec{v}|$)

$$\boxed{\vec{a}_t = \frac{dv}{dt}\vec{\tau}}$$

→ **aceleración normal**: relacionada con la variación de la dirección de la velocidad ($\vec{v}$).

$$\boxed{\vec{a}_n = v\frac{d\vec{\tau}}{dt}}$$

- Momento lineal (cantidad de movimiento)

$$\boxed{\vec{p} = m\vec{v}}$$

- Momento angular o cinético

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

Tema 1 Mecánica de Newton

1.1 Sistemas de referencia inerciais. Leis de Newton

Un sistema de referencia inercial é aquel no que se cumplen as leis de Newton do movemento. Un sistema non pode ser inercial se describe un movemento acelerado respecto de outro que si o sexa.

Leis de Newton

- 1ª Lei (inercia): ⟶ define { - forza cero / - sistema inercial

Un corpo móvese a velocidade constante a non ser que actúe unha forza sobre el.

- 2ª Lei (lei fundamental da dinámica): ⟶ define forza non nula

$$\vec{p} = m\vec{v} \qquad \boxed{\frac{d\vec{p}}{dt} = \vec{F}} \underset{m\ \text{cte}}{\Longrightarrow} \boxed{m\vec{a} = \vec{F}}$$

Ecuación de movemento de Newton: $\boxed{m\frac{d^2\vec{r}}{dt^2} = \vec{F}}$

- 3ª Lei (acción e reacción):

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}$$

Conservación do momento lineal:

$$\frac{d\vec{p}}{dt} = \frac{d(m_1\vec{v}_1 + m_2\vec{v}_2)}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = m_1\vec{a}_1 + m_2\vec{a}_2 =$$

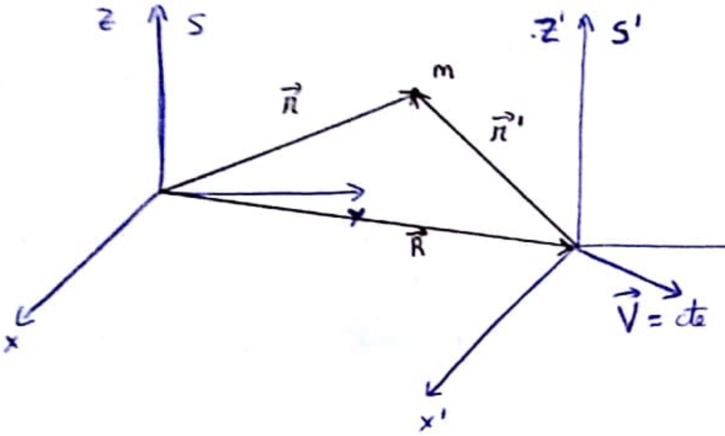
$$\underset{2^a\ \text{Lei}}{=} \vec{F}_{21} + \vec{F}_{12} \underset{3^a\ \text{Lei}}{=} 0 \Rightarrow \frac{d\vec{p}}{dt} = 0 \Rightarrow \boxed{\vec{p} = \text{cte}}$$

Transformacións de Galileo:

(transformación de coordenadas entre sistemas inerciais):

- Principio de relatividade de Galileo

Dous sistemas de referencia cuxo movemento relativo consiste nunha traslación rectilínea uniforme son equivalentes desde o punto de vista da mecánica (as leis mecánicas coinciden).



S' móvese con $\boxed{\vec{V} = cte}$ respecto de S

$$\vec{r} = \vec{r}\,' + \vec{R}$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}\,'}{dt} + \frac{d\vec{R}}{dt}$$

$$\vec{v} = \vec{v}\,' + \vec{V}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}\,'}{dt^2} + \frac{d^2\vec{R}}{dt^2} = 0 \quad (\vec{V} = cte)$$

$$\boxed{\vec{a} = \vec{a}\,'}$$

$$\vec{F} = m\vec{a} = m\vec{a}\,' = \vec{F}\,'$$

$$\boxed{\vec{F} = \vec{F}\,'}$$

$$\vec{V} = (V_x, V_y, V_z)$$

Aceleración invariante

Transformadas de Galileo:

$$\begin{cases} x = x' + V_x t \\ y = y' + V_y t \\ z = z' + V_z t \\ t = t' \quad (\text{tempo é absoluto}) \end{cases}$$

Ppio de Galileo: todas as leis da mecánica son invariantes fronte ás transformacións de Galileo.

→ Só válido na mecánica clásica

1.2 Teoremas de conservación

1.2.1 Dinámica dunha partícula

(A) Conservación do momento lineal

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}} \quad , \quad \boxed{\vec{F} = 0} \Rightarrow \frac{d\vec{p}}{dt} = 0 \Rightarrow \boxed{\vec{p} = m\vec{v} = cte}$$

Se non hai forza que actúe sobre a partícula o momento lineal consérvase.

(B) Conservación do momento angular: $\boxed{\vec{L} = \vec{r} \times \vec{p}}$

- Definimos o Torque ou momento dunha forza respecto dun punto como : $\boxed{\vec{M} = \vec{r} \times \vec{F}}$

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} =$$

$$= \cancel{\vec{v} \times m\vec{v}}^{\,0} + \vec{r} \times \vec{F} = \vec{M}$$

$(\vec{v} \parallel m\vec{v})$

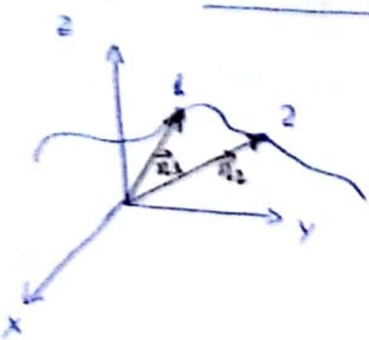
$$\boxed{\frac{d\vec{L}}{dt} = \vec{M}} \qquad \boxed{\vec{M} = 0 \Rightarrow \vec{L} = cte}$$

Se o momento da forza é nulo consérvase o momento angular.

Isto sucede cando $\vec{r} = 0$, $\vec{F} = 0$ ou $\vec{r} \parallel \vec{F}$ (p.e. forzas centrais).

$$\vec{L} = cte \Rightarrow \vec{r} \times \vec{p} = m\underbrace{(\vec{r} \times \vec{v})}_{\vec{r}\text{ e }\vec{v}\text{ teñen que ser coplanares}} = cte$$

(c) Conservación da enerxía



O traballo realizado por unha forza sobre unha partícula equivale á enerxía necesaria para desprazala

$$W_{1\to2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (F_x dx + F_y dy + F_z dz)$$

Definimos a <u>enerxía cinética (T)</u> como a enerxía que posúe un corpo debida ao seu movemento: $\boxed{T = \frac{1}{2} m \vec{v}^2}$

$$W_{1\to2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{d\vec{p}}{dt}\cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} m \frac{d\vec{v}}{dt}\cdot d\vec{r} =$$

$$= m \int_{\vec{v}_1}^{\vec{v}_2} \vec{v}\cdot d\vec{v} = m \left(\frac{\vec{v}^2}{2}\right]_{\vec{v}_1}^{\vec{v}_2} = \frac{1}{2} m (v_2^2 - v_1^2) =$$

$$= T_2 - T_1$$

Acabamos de demostrar que o traballo coincide coa variación da enerxía cinética entre dous puntos $\boxed{W_{1\to2} = T_2 - T_1}$
(Th da E cinética)

Unha <u>forza</u> é <u>conservativa</u> se o traballo realizado para mover unha partícula é independente da traxectoria.

$$\int_c \vec{F}\cdot d\vec{r} = cte \implies \oint_c \vec{F}\cdot d\vec{r} = 0$$

<u>Th.</u> Se unha forza $\underbrace{\vec{F}(\vec{r})}_{\text{é conservativa}}$ existe un potencial $V(\vec{r})$ tal que:

$$\boxed{\vec{F}(\vec{r}) = -\vec{\nabla} V(\vec{r})} \qquad \vec{F} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

Se $\vec{F}$ é conservativa :

$$W_{1\to2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla} V \cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} \frac{dV}{d\vec{r}} \cdot d\vec{r} =$$

$$= -\int_{\vec{r}_1}^{\vec{r}_2} dV = -\left(V(\vec{r})\right]_{\vec{r}_1}^{\vec{r}_2} = -\left(V_2 - V_1\right)$$

Así demostramos que para forzas conservativas o traballo para mover unha partícula é igual á menos variación de enerxía potencial entre dous puntos: $\boxed{W_{1\to2} = -(V_2 - V_1)}$

(Th. da E potencial)

$$\left.\begin{array}{l} W_{1\to2} = T_2 - T_1 \\ W_{1\to2} = -(V_2 - V_1) \end{array}\right\} \Rightarrow T_2 - T_1 = V_1 - V_2$$

$$\boxed{T_1 + V_1 = T_2 + V_2}$$

$\downarrow$

Á suma de enerxías potencial e cinética chamámoslle enerxía mecánica

$$\boxed{E = T + V = cte}$$

Teorema de conservación da enerxía mecánica

(Forzas conservativas)

$$\left| \begin{array}{l} \vec{F} = -\vec{\nabla} V \\ \vec{\nabla} \times \vec{F} = 0 \end{array}\right.$$

1.2.2 Dinámica de Sistemas de Partículas

$\boxed{N \text{ partículas}}$

Notación → Fuerzas entre partículas:

$\vec{F}_{ij}$ fuerza de la partícula i sobre la partícula j.

→ $\vec{F}_i^{(e)}$ fuerza externa sobre la partícula i

Para cada partícula: $\boxed{\dfrac{d\vec{p}_i}{dt} = \vec{F}_i^{(e)} + \sum_{j\neq i}^{N} \vec{F}_{ji}}$

(A) Conservación do momento lineal total:

Para cada parella de partículas: $\boxed{\vec{F}_{ij} = -\vec{F}_{ji}}$
(3ª Lei Newton)

Se $\boxed{\sum_{i=1}^{N} \vec{F}_i^{(e)} = 0} \Rightarrow \sum \vec{F} = 0 = \dfrac{d\vec{p}}{dt}$

$\Downarrow$

$\boxed{\vec{p} = \sum \vec{p}_i = \text{cte}}$

$$\sum \vec{F}_i^{(e)} + \sum_i \sum_j \overset{\nearrow 0}{\vec{F}_{ji}} = \frac{d\vec{p}}{dt}$$

$(\vec{F}_{ji} = -\vec{F}_{ij})$

(*) Se a resultante das forzas externas é nula, consérvase o momento lineal total.

(B) Conservación do momento angular total:

$\vec{F}_{ij} = -\vec{F}_{ji}$

$(\vec{r}_i - \vec{r}_j) \parallel \vec{F}_{ji}$

Se $\sum \vec{M}_i^{(e)} = \sum \vec{r}_i \times \vec{F}_i^{(e)} = 0 \Rightarrow \dfrac{d\vec{L}}{dt} = 0$

$\Downarrow$

$\boxed{\vec{L} = \text{cte}}$

$$\frac{d\vec{L}}{dt} = \sum \vec{M}_i^{(e)} + \cancel{\sum_i \sum_j \vec{r}_i \times \vec{F}_{ji}}^{\,0} = \sum \vec{M}_i^{(e)} = \vec{M}^{(e)}$$

$\boxed{\sum \vec{L}_i = \text{cte}}$

$$(\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij}) = \vec{r}_i \times \vec{F}_{ji} - \vec{r}_j \times \vec{F}_{ji} =$$
$$= (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = 0$$

(*) Se o momento das forzas externas é nulo consérvase o momento angular total.

Ⓒ Conservación da enerxía:

Ⓘ Enerxía cinética $\longrightarrow$ $T = \frac{1}{2}\sum_i m_i \vec{v}_i^{\,2}$

$$W_{1\to2} = \sum_i \int_1^2 \vec{F}_i \, d\vec{r}_i = \int_1^2 dT = T_2 - T_1$$

↳ Traballo para trasladar o sistema desde 1 a 2

Ⓘⓘ Supoñamos que as forzas externas son conservativas: $\vec{F}_i^{(e)} = -\vec{\nabla}_i V_i^{(e)}$

$$W_{1\to2}^{(e)} = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{r}_i = -\sum_i \int_1^2 \vec{\nabla}_i V_i^{(e)} \cdot d\vec{r}_i =$$

$$= -\sum_i \int_1^2 dV_i^{(e)} = -\int_1^2 dV^{(e)} = -\left(V_2^{(e)} - V_1^{(e)}\right) = V_1^{(e)} - V_2^{(e)}$$

Ⓘⓘⓘ Supoñamos as forzas internas tamén conservativas: $\vec{F}_{ji} = -\vec{\nabla}_i V_{ji}$

$\vec{F}_{ij} = -\vec{\nabla}_j V_{ij}$

$$W_{1\to2}^{int} = \sum_i \sum_j \int_1^2 \vec{F}_{ji} \cdot d\vec{r}_i = \frac{1}{2} \sum_i \sum_j \int_1^2 \left(\vec{F}_{ji} \cdot d\vec{r}_i + \vec{F}_{ij} \cdot d\vec{r}_j\right) =$$

$$= -\frac{1}{2} \sum_i \sum_j \int_1^2 \left(\vec{\nabla}_i V_{ji} \cdot d\vec{r}_i + \vec{\nabla}_j V_{ij} \cdot d\vec{r}_j\right) =$$

$$= -\frac{1}{2} \sum_i \sum_j \int_1^2 dV_{ij} = -\int_1^2 dV^{(int)} = V_1^{(int)} - V_2^{(int)}$$

$$\begin{cases} V^{(i)} = \frac{1}{2} \sum_i \sum_j V_{ij} \\ V^{(e)} = \sum_i V_i^{(e)} \end{cases}$$

$$\begin{cases} W_{1\to2} = \Delta T \\ W_{1\to2} = -\Delta V^{(e)} - \Delta V^{(i)} \\ \quad \Delta T + \Delta V^{(e)} + \Delta V^{(i)} = 0 \\ \quad \Delta\left(T + V^{(e)} + V^{(i)}\right) = 0 \end{cases}$$

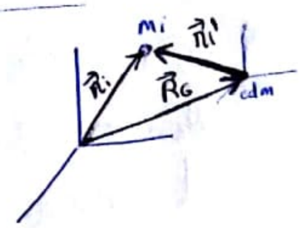
Th. Conservación da enerxía (forzas ext e int. conservativas)

$$\longrightarrow \boxed{T + V^{(e)} + V^{(i)} = cte = E}$$

$$\boxed{E = \sum T_i + \sum V_i^{(e)} + \frac{1}{2}\sum_i\sum_j V_{ij}}$$

⊛ Variables do centro de masas (c.d.m.):

$$\boxed{\vec{R}_G = \frac{\sum_i m_i \cdot \vec{r}_i}{\sum_i m_i}} \qquad M = \sum_i m_i$$



$$\boxed{\vec{V}_G = \frac{d\vec{R}_G}{dt} = \frac{\sum_i m_i \vec{v}_i}{M}}$$

$$\rightsquigarrow \vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i = \sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right) =$$
$$= \frac{d}{dt}\left(M \cdot \vec{R}_G\right) = M \cdot \frac{d\vec{R}_G}{dt} = M \cdot \vec{V}_G$$

Ⓘ $\boxed{\vec{P} = \vec{P}_G = M \cdot \vec{V}_G} \longrightarrow$ momento lineal (Th. 0 König)

$$\rightsquigarrow \vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \left[\left(\vec{r}_i' + \vec{R}_G\right) \times m_i\left(\vec{v}_i' + \vec{V}_G\right)\right] =$$
$$= \sum_i \vec{r}_i' \times m_i \vec{v}_i' + \sum_i \vec{R}_G \times m_i \vec{V}_G +$$
$$+ \underbrace{\sum_i \vec{r}_i' \times m_i \vec{V}_G}_{\to 0} + \underbrace{\sum_i \vec{R}_G \times m_i \vec{v}_i'}$$

$$\sum_i \vec{r}_i' \times m_i \vec{V}_G = \sum_i m_i \vec{r}_i' \times \vec{V}_G = M \cancelto{0}{\vec{r}_G'} \times \vec{V}_G = 0$$

$$\sum_i \vec{R}_G \times m_i \vec{v}_i' = \vec{R}_G \times \sum_i m_i \vec{v}_i' = \vec{R}_G \times \sum_i m_i \frac{d\vec{r}_i'}{dt} =$$
$$= \vec{R}_G \times \frac{d}{dt}\left(\sum_i m_i \vec{r}_i'\right) = \vec{R}_G \times \frac{d}{dt}\left(M \cdot \cancelto{0}{\vec{r}_G'}\right) = 0$$

$$\vec{L} = \sum_i \vec{r}_i' \times m_i \vec{v}_i' + \vec{R}_G \times M\vec{V}_G$$

Ⓘⓘ $\boxed{\vec{L} = \vec{L}' + \vec{L}_G} \longrightarrow$ momento angular (1er Th. König)

$\vec{L}$: Momento angular respecto dun punto fixo

$\vec{L}'$: Momento angular respecto do c.d.m.

$\vec{L}_G$: Momento angular respecto do punto fixo do vector cantidade de movemento do sistema situado no c.d.m.

$$\rightsquigarrow \quad T = \sum_i T_i = \frac{1}{2}\sum_i m_i \vec{v}_i^{\,2} = \frac{1}{2}\sum_i m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 =$$

$$= \frac{1}{2}\sum_i m_i \left(\frac{d(\vec{r}_G + \vec{r}_i')}{dt}\right)^2 = \frac{1}{2}\sum_i m_i \left[\frac{d\vec{r}_G}{dt} + \frac{d\vec{r}_i'}{dt}\right]^2 =$$

$$= \frac{1}{2}\sum_i m_i \left[\vec{v}_G + \vec{v}_i'\right]^2 = \frac{1}{2}\sum_i m_i \vec{v}_G^{\,2} +$$

$$+ \frac{1}{2}\sum_i m_i \vec{v}_i'^{\,2} + \sum_i m_i \vec{v}_i' \vec{v}_G =$$

$$= \underbrace{\frac{1}{2} M \vec{v}_G^{\,2}}_{T_G} + \underbrace{\frac{1}{2}\sum_i m_i \vec{v}_i'^{\,2}}_{T'} + (M \cancel{\vec{v}_G'}) \cdot \vec{v}_G$$

$$\boxed{T = T_G + T'} \longrightarrow \text{enerxía cinética} \quad (2^{\circ}\ \text{Th. König})$$

- T: Enerxía cinética do sistema
- T_G: Enerxía cinética do c.d.m dotado de toda a masa
- T': Enerxía cinética debida ao movemento en torno ao c.d.m.

1.3 Exemplos de integración das ecuacións de Newton

$$\boxed{m \frac{d^2\vec{r}}{dt^2} = \vec{F}(t, \vec{r}, \vec{v})}$$

Ecuación da 2ª lei de Newton

Traballamos nunha dimensión:

$$\vec{r} \longrightarrow x \qquad \vec{v} \longrightarrow v \qquad \vec{F} \longrightarrow F$$

1 Dimensión espacial
$m = cte$

$$\boxed{m \frac{d^2 x}{dt^2} = F(t, x, v)}$$

Forzas dependentes

- do tempo $\quad m \dfrac{dv}{dt} = F(t) \qquad$ p.e. $F(t) = F_0 \cos(\omega t + \theta)$
- da velocidade $\quad m \dfrac{dv}{dt} = F(v) \qquad F(v) = -bv$
- da posición $\quad m \dfrac{dv}{dt} = F(x) \qquad F(x) = -kx$

Ⓐ Forzas dependentes do tempo:

$$\boxed{m\,\frac{dv}{dt} = F(t)}$$

Ecuación diferencial de segunda orde

$\hookrightarrow$ 2 integrais e 2 ctes de integración

$$m\,dv = F(t)\,dt\,,\quad m\int_{v_0}^{v} dv = \int_{t_0}^{t} F(t')\,dt'$$

$$m\,(v - v_0) = \int_{t_0}^{t} F(t')\,dt'$$

$$v = \frac{dx}{dt} \longleftarrow \boxed{v(t) = v_0 + \frac{1}{m}\underbrace{\int_{t_0}^{t} F(t')\,dt'}_{I(t)}}$$

$$\int_{x_0}^{x} dx' = \int_{t_0}^{t}\left(v_0 + \frac{1}{m}\int_{t_0}^{t''} F(t')\,dt'\right) dt''$$

$$x(t) - x_0 = v_0\int_{t_0}^{t} dt'' + \frac{1}{m}\int_{t_0}^{t}\left(\int_{t_0}^{t''} F(t')\,dt'\right) dt''$$

$$\boxed{x(t) = x_0 + v_0\,(t - t_0) + \frac{1}{m}\int_{t_0}^{t}\underbrace{\left(\int_{t_0}^{t'} F(t'')\,dt''\right)}_{I(t')} dt'}$$

p.e. $F(t) = mg$

$$m\,\frac{d^2x}{dt^2} = mg\,,\quad \frac{d^2x}{dt^2} = g$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = g\,,\quad \frac{dx}{dt} = g\int_{t_0}^{t} dt + C\ (= v_0)$$

$$\frac{dx}{dt} = v(t) = g\,(t - t_0) + v_0$$

$$x(t) = \int_{t_0}^{t}\left[g(t - t_0) + v_0\right] dt + \underset{x_0}{\overset{\|}{C'}}\,,\quad \boxed{x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}g(t - t_0)^2}$$

Outro exemplo: $\boxed{F(t) = \underset{\text{amplitude}}{F_o} \cdot \cos(\underset{\text{frecuencia}}{\omega} t + \underset{\text{fase}}{\theta})}$ → Forza oscilatoria (o ángulo da oscilación depende do tempo)

$$I(t') = \int_{t_o}^{t'} F(t'')\,dt'' = F_o \int_{t_o}^{t'} \cos(\omega t'' + \theta)\,dt'' = \frac{F_o}{\omega}\left(\sin(\omega t'' + \theta)\right]_{t_o}^{t'} =$$

$$= \frac{F_o}{\omega}\left(\sin(\omega t' + \theta) - \sin(\omega t_o + \theta)\right)$$

$$v(t) = v_o + \frac{1}{m}\int_{t_o}^{t} F(t')\,dt' \qquad \boxed{v(t) = v_o + \frac{F_o}{m\omega}\left[\sin(\omega t + \theta) - \sin(\omega t_o + \theta)\right]}$$

Comprobemos que isto ten unidades de velocidade
→ senos adimensionales
→ $\left[\frac{F_o}{m\omega}\right] = \frac{M\,L/T^2}{M/T} = \frac{L}{T}$ ✓

$$\int_{t_o}^{t} I(t')\,dt' = \int_{t_o}^{t} \frac{F_o}{\omega}\left(\sin(\omega t' + \theta) - \sin(\omega t_o + \theta)\right)dt' =$$

$$= \frac{F_o}{\omega}\left[\left(-\frac{\cos(\omega t' + \theta)}{\omega}\right]_{t_o}^{t} - \sin(\omega t_o + \theta)(t - t_o)\right] =$$

$$= \frac{F_o}{\omega^2}\left[\cos(\omega t_o + \theta) - \cos(\omega t + \theta) - \omega \sin(\omega t_o + \theta)(t - t_o)\right]$$

$$x(t) = x_o + v_o(t - t_o) + \frac{1}{m}\int_{t_o}^{t} dt' \int_{t_o}^{t'} F(t'')\,dt''$$

$$\boxed{x(t) = x_o + v_o(t - t_o) + \frac{F_o}{m\omega^2}\left[\cos(\omega t_o + \theta) - \cos(\omega t + \theta) - \omega \sin(t_o\omega + \theta)(t - t_o)\right]}$$

Fixamos: c.i. $\begin{cases} t_o = 0 \\ x_o = 0 \\ v_o = 0 \end{cases}$ ⟶ $\boxed{x(t) = \frac{F_o}{m\omega^2}\left[\cos\theta - \cos(\omega t + \theta) - \omega t \sin\theta\right]}$

Con tempos moi pequenos ($t \to 0$, $t \ll 1$): $F(t) \approx F_o \cos\theta$

Desenvolvemento de Taylor en torno a $t = 0$: $f(t) = f(0) + \frac{1}{1!}\left.\frac{df}{dt}\right|_{t=0} t + \frac{1}{2!}\left.\frac{d^2 f}{dt^2}\right|_{t=0} t^2 + \dots$

$f(t) = \cos(\omega t + \theta) \qquad f(0) = \cos\theta$

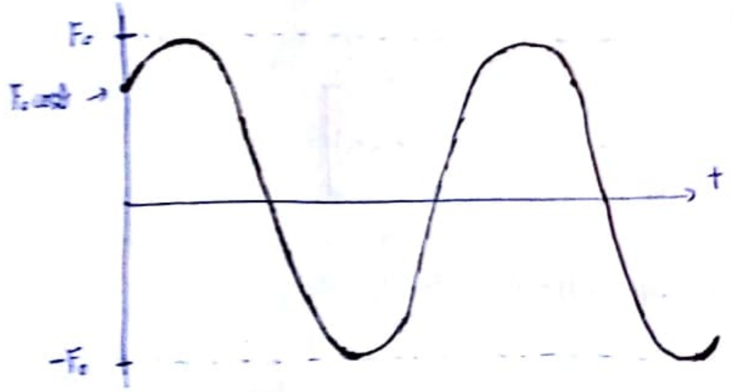
$\frac{d}{dt}(\cos(\omega t + \theta)) = -\omega \sin(\omega t + \theta)\ , \quad f'(t=0) = -\omega \sin\theta$

$\frac{d^2}{dt^2}(\cos(\omega t + \theta)) = -\omega^2 \cos(\omega t + \theta)\ , \quad f''(t=0) = -\omega^2 \cos\theta$

Taylor 2º orden: $\boxed{\cos(\omega t + \theta) \approx \cos\theta - \omega t \sin\theta - \frac{1}{2}\omega^2 t^2 \cos\theta}$

$x(t) = \frac{F_0}{m\omega^2}\left[\cancel{\cos\theta} - \cancel{\cos\theta} + \cancel{\omega t \sin\theta} + \frac{1}{2}\omega^2 t^2 \cos\theta - \cancel{\omega t \sin\theta}\right]$

$$\boxed{x(t) = \frac{F_0}{2m}\cos\theta\ t^2}$$



$\boxed{F(t) = F_0 \cos(\omega t + \theta)}$

* Con tiempos mui grandes $\longrightarrow$ $\boxed{x(t) \longrightarrow -\frac{F_0}{m\omega}\sin\theta\ t}$
$(t \to \infty)$
$(t \gg 1)$

* Con $\boxed{\theta = 0}$ $\to$ $\boxed{x(t) = \frac{F_0}{m\omega^2}\left[1 - \cos(\omega t)\right]}$

$0 \le x(t) \le 2 \cdot \frac{F_0}{m\omega^2}$

(B) Forzas dependentes da velocidade:

$$\boxed{m\frac{dv}{dt} = F(v)}$$

$$m\int_{v_0}^{v}\frac{dv}{F(v)} = \int_{t_0}^{t} dt \longrightarrow G(v) - G(v_0) = \frac{t-t_0}{m}$$

$$\frac{dx}{dt} = \boxed{v(t) = f\left(v_0, \frac{t-t_0}{m}\right)}$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} f\left(v_0, \frac{t-t_0}{m}\right) dt \;, \quad \boxed{x(t) = x_0 + \int_{t_0}^{t} f\left(v_0, \frac{t'-t_0}{m}\right) dt'}$$

p.e. $\boxed{F(v) = -b\,v^n} \longrightarrow$ Forza de rozamento

$\boxed{n=1} \longrightarrow \boxed{F(v) = -bv}$

$$[b] = \frac{M\,L/T^2}{L/T} = \frac{M}{T}$$

$$m\frac{dv}{dt} = -bv$$

$$m\int_{v_0}^{v}\frac{dv'}{v'} = -b\int_{t_0}^{t} dt'$$

$$(\ln v']_{v_0}^{v} = -\frac{b}{m}(t']_{t_0}^{t}$$

$$\ln v - \ln v_0 = -\frac{b}{m}(t-t_0)$$

$$\ln\frac{v}{v_0} = -\frac{b}{m}(t-t_0)$$

$$\boxed{v(t) = v_0 \cdot e^{-\frac{b}{m}(t-t_0)}}$$

$$\frac{dx}{dt} = v$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} v \cdot dt$$

$$x = x_0 + \int_{t_0}^{t} v_0 \cdot e^{-\frac{b}{m}(t-t_0)} dt$$

$$x(t) = x_0 + v_0\left(\frac{e^{-\frac{b}{m}(t-t_0)}}{-b/m}\right]_{t_0}^{t} = x_0 + v_0\frac{m}{b}\left(1 - e^{-\frac{b}{m}(t-t_0)}\right)$$

$$\boxed{x(t) = x_0 + v_0\frac{m}{b}\left(1 - e^{-\frac{b}{m}(t-t_0)}\right)}$$

c.i. $\begin{cases} t_0 = 0 \\ x_0 = 0 \\ v_0 \neq 0 \end{cases}$

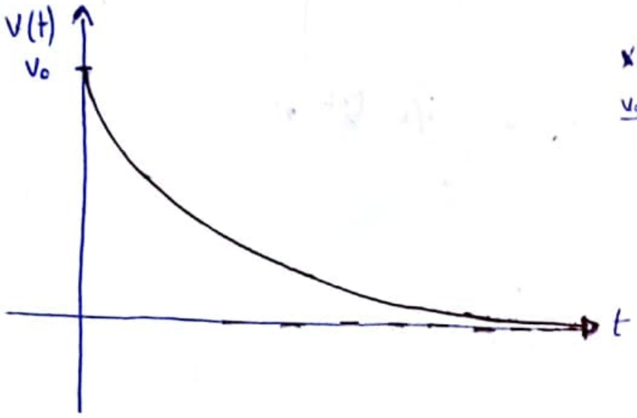
$$v(t) = v_0 \, e^{-\frac{b}{m}t}$$

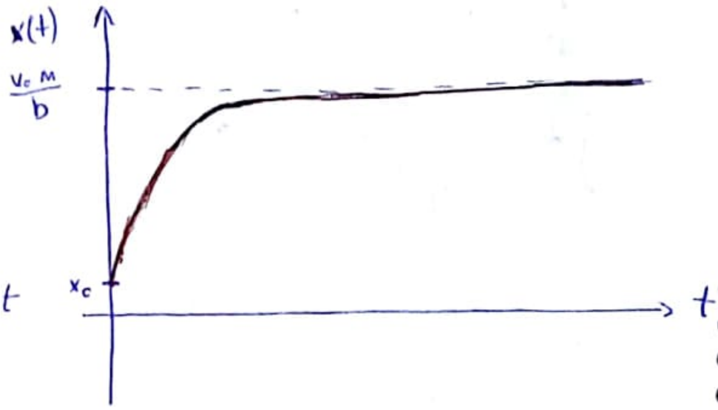
$$x(t) = \frac{v_0 m}{b}\left(1 - e^{-\frac{b}{m}t}\right)$$

* Cando $t \to \infty$, $x(t) \longrightarrow x_0 + \frac{m}{b} v_0$

* Cando $t \to 0$, $x(t) = v_0 t - \frac{1}{2} \frac{v_0 b}{m} t^2$ (Taylor)

Gráficas:





Ⓒ Forzas dependentes da posición:

$F = F(x)$

$$\boxed{m \frac{dv}{dt} = F(x)}$$

→ As forzas dependentes únicamente da posición en $\boxed{\text{unha dimensión}}$ son conservativas.

$\Downarrow$

$\exists (V(x)) /$ → $V(x) = -\int F(x)\,dx + cte$

→ $F(x) = -\frac{dV}{dx}$

$$m \frac{dv}{dx} \cdot \frac{dx}{dt} = F(x)$$

$$m \int_{v_0}^{v} v' \cdot dv' = \int_{x_0}^{x} F(x) \cdot dx$$

$$m \left(\frac{v^2}{2}\right]_{v_0}^{v} = -\left(V(x')\right]_{x_0}^{x}$$

$$\frac{1}{2} m v^2 - \frac{1}{2} m v_0^2 = V(x_0) - V(x)$$

* Se as integrais fosen indefinidas:

$$\frac{1}{2} m v^2 + cte = -V + cte'$$

$$\boxed{\frac{1}{2} m v^2 + V} = cte' - cte = \boxed{E}$$

→ a enerxía é a cte de integración

conservación da enerxía mecánica

$$\boxed{\frac{1}{2} m v^2 + V(x) = \frac{1}{2} m v_0^2 + V(x_0) = cte = E}$$

$$E = \frac{1}{2} m v^2 + V$$

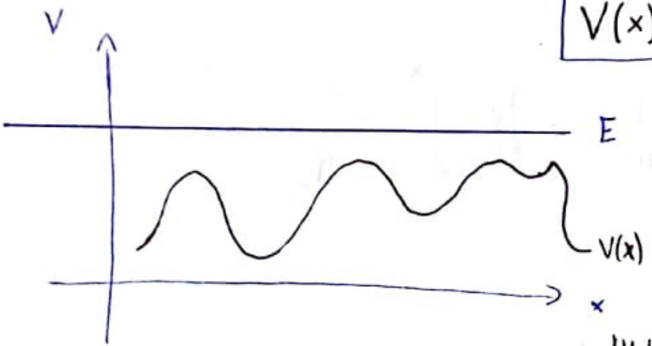
$$\boxed{v = \pm\sqrt{\frac{2}{m}\left(E - V(x)\right)}} = \frac{dx}{dt}$$

$$\int_{x_0}^{x} \frac{dx'}{\sqrt{E - V(x)}} = \int_{t_0}^{t} \pm\sqrt{\frac{2}{m}}\, dt' \quad , \quad F(x) - F(x_0) = \sqrt{\frac{2}{m}}\,(t - t_0)$$

$$\boxed{x(t) = \varphi(x_0, t)}$$

Gráficas: ⊛ Nota: A enerxía potencial non pode superar nunca á enerxía total:

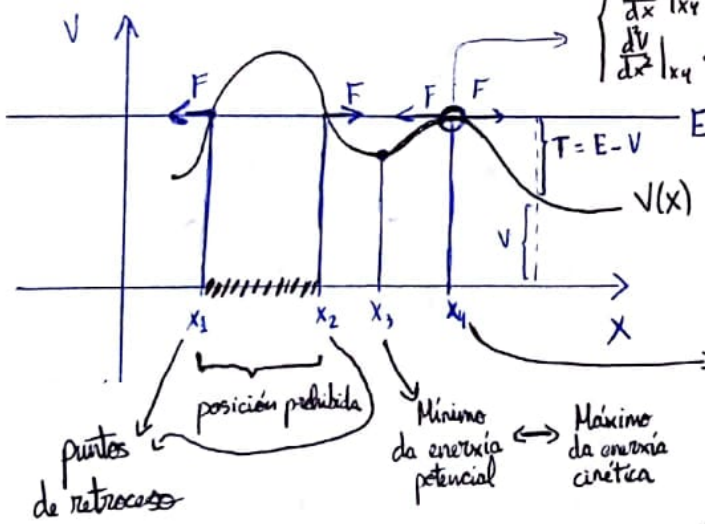
$$\boxed{V(x) \leq E}$$



$$\begin{cases} \left.\frac{dV}{dx}\right|_{x_4} = 0 \Rightarrow F(x_0) = -\left.\frac{dV}{dx}\right|_{x_4} = 0 \\ \left.\frac{d^2V}{dx^2}\right|_{x_4} < 0 \Rightarrow \left.\frac{dF}{dx}\right|_{x_4} = -\left.\frac{d^2V}{dx^2}\right|_{x_4} > 0 \end{cases}$$

$\boxed{\left.\frac{dF}{dx}\right|_{x_4} > 0} \Rightarrow$ dF e dx teñen o mesmo signo

↳ A forza actúa desprazándose de x_4



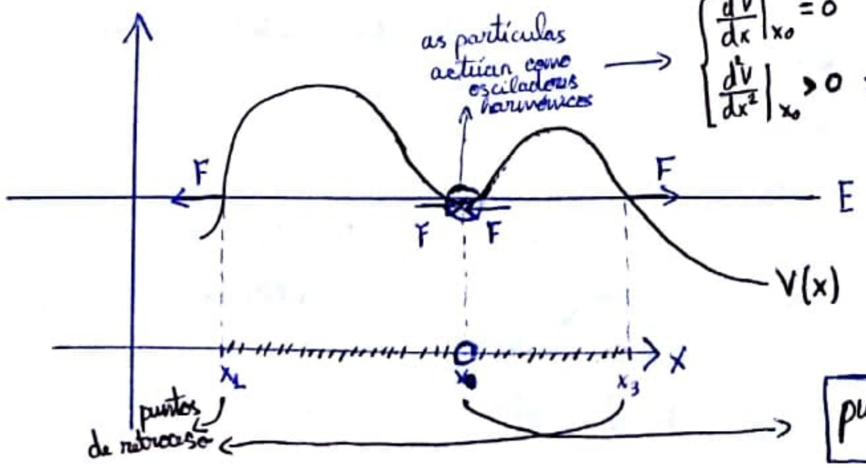
$$\boxed{\begin{array}{c}\text{punto de equilibrio inestable} \\ (E_{p_{máx}} = E_{total})\end{array}}$$

$$\begin{cases} \left.\frac{dV}{dx}\right|_{x_0} = 0 \Rightarrow F(x_0) = -\left.\frac{dV}{dx}\right|_{x_0} = 0 \\ \left.\frac{d^2V}{dx^2}\right|_{x_0} > 0 \Rightarrow \left.\frac{dF}{dx}\right|_{x=x_0} = -\left.\frac{d^2V}{dx^2}\right|_{x=x_0} < 0 \end{cases}$$

$\boxed{\left.\frac{dF}{dx}\right|_{x_0} < 0} \Rightarrow$ dF e dx teñen signo oposto

↓

A forza actúa cara ao centro da oscilación (x_0)



$$\boxed{\begin{array}{c}\text{punto de equilibrio estable} \\ (E_{p_{mín}} = E_{total})\end{array}}$$

p.e. $\boxed{F(x) = -kx}$ Lei de Hooke

$$V(x) = -\int F(x)\cdot dx = \int kx \cdot dx = \frac{kx^2}{2} + C$$

$\circledast$ $\boxed{C=0} \longrightarrow \quad V(x) = \frac{1}{2}kx^2$ c.i. $\begin{cases} t_0 = 0 \\ x_0 = 0 \rightarrow V(x_0) = 0 \end{cases}$

$$E = T + V = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{dx}{dt} = v = \sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2\right)} \longrightarrow \boxed{\int_0^x \frac{dx}{\sqrt{E - \frac{1}{2}kx^2}} = \int_0^t \sqrt{\frac{2}{m}}\,dt}$$

$$\int_0^x \frac{dx}{\sqrt{E-\frac{1}{2}kx^2}} = \int_0^x \frac{1}{\sqrt{1-\frac{kx^2}{2E}}}\frac{dx}{\sqrt{E}} = \sqrt{\frac{2}{k}}\int_0^x \frac{dx}{\sqrt{1-\left(\sqrt{\frac{k}{2E}}x\right)^2}}\sqrt{\frac{k}{2E}} =$$

$$= \sqrt{\frac{2}{k}}\arcsin\left(\sqrt{\frac{k}{2E}}x\right)$$

$$\int_0^t \sqrt{\frac{2}{m}}dt = \sqrt{\frac{2}{m}}t$$

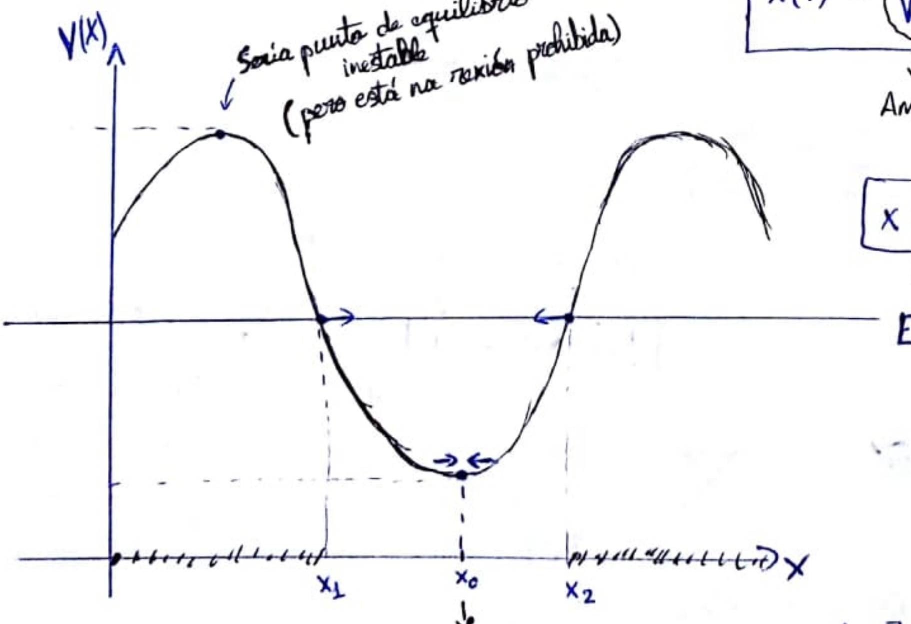
$$\sqrt{\frac{2}{k}}\arcsin\left(\sqrt{\frac{k}{2E}}x\right) = \sqrt{\frac{2}{m}}t$$

$$\arcsin\left(\sqrt{\frac{k}{2E}}x\right) = \sqrt{\frac{k}{m}}t$$

$$\sqrt{\frac{k}{2E}}x = \sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$\boxed{x(t) = \underbrace{\sqrt{\frac{2E}{k}}}_{\substack{\text{Amplitude} \\ A}}\cdot \sin\Big(\underbrace{\sqrt{\frac{k}{m}}}_{\substack{\text{Velocidade angular} \\ \omega}}t\Big)}$$

$$\boxed{x(t) = A\sin(\omega t)} \rightarrow \text{movemento oscilatorio}$$



Punto de equilibrio estable $\quad F(x_0) = -\frac{dV}{dx}\Big]_{x_0} = 0$

+++++++ Rexión prohibida

$(V(x) > E)$

(*) Para oscilacións moi próximas a x_0 :

$$V(x) = V(x_0) + \cancel{\left.\frac{dV}{dx}\right|_{x_0}}^{\,0} (x-x_0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_0} (x-x_0)^2 + \dots$$

$V(x) = \frac{kx^2}{2}$ $\qquad \left.\frac{dV}{dx}\right|_{x_0} = 0$ (x_0 é un mínimo)

$V'(x) = kx$

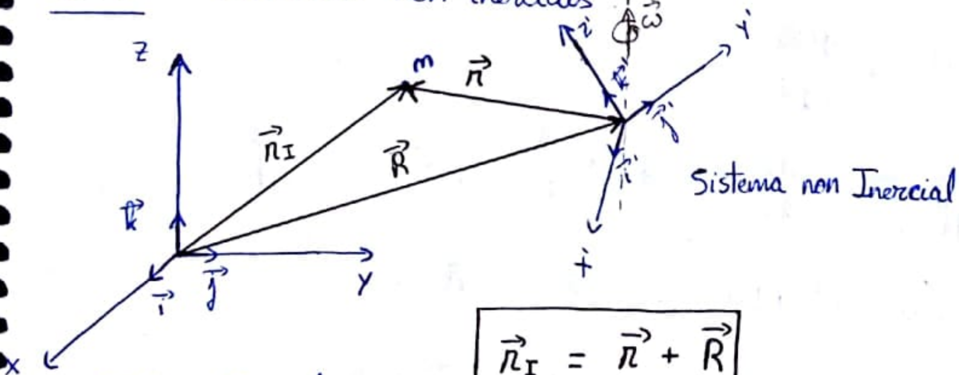
$V''(x) = k$

$$\boxed{V(x) = V(x_0) + \frac{1}{2} k (x-x_0)^2}$$

→ Oscilador harmónico

Cando $x \to 0$: $\qquad$ M.A.S. $\longrightarrow$ $\boxed{\begin{array}{c} x(t) = A \cdot \sin(\omega t + \varphi) \\ \omega \simeq \sqrt{\frac{k}{m}} \end{array}}$

1.4 Sistemas non inerciais :



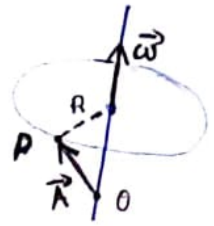
$$\boxed{\vec{r}_I = \vec{r}' + \vec{R}}$$

Nos sistemas de referencia non inerciais non se cumplen as leis de Newton, sendo $\vec{F} \neq m\vec{a}$. Obteremos unha nova ecuación.

$$\vec{F} = m \cdot \vec{a}_I = m \cdot \frac{d^2\vec{r}_I}{dt^2} = m \cdot \frac{d^2(\vec{r}+\vec{R})}{dt^2} = m\frac{d^2\vec{r}}{dt^2} + m\frac{d^2\vec{R}}{dt^2}$$

$$\boxed{m\frac{d^2\vec{r}}{dt^2} = \vec{F} - m\frac{d^2\vec{R}}{dt^2}}$$

(*) Sexa $\vec{A}$ un vector de módulo constante que rota arredor dun eixo con $\vec{\omega} = cte$



$|\vec{A}| = cte \Rightarrow$ P describe unha circunferencia

$$\left.\begin{aligned} \vec{v}_P &= \frac{d\vec{A}}{dt} \\ \vec{v}_P &= \vec{\omega} \times \vec{A} \end{aligned}\right\} \Rightarrow \boxed{\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}}$$

Se temos unha base vectorial con orixe en O (como a do noso S.R. non inercial):

$$\begin{cases} \dfrac{d\vec{i}'}{dt} = \vec{\omega} \wedge \vec{i}' \\ \dfrac{d\vec{j}'}{dt} = \vec{\omega} \wedge \vec{j}' \\ \dfrac{d\vec{k}'}{dt} = \vec{\omega} \wedge \vec{k}' \end{cases}$$

Ecuacións de Poisson

$$\vec{r} = x\vec{i}' + y\vec{j}' + z\vec{k}'$$

$$\frac{d\vec{r}}{dt} = \frac{d(x\vec{i}')}{dt} + \frac{d(y\vec{j}')}{dt} + \frac{d(z\vec{k}')}{dt} =$$

$$= \underbrace{\frac{dx}{dt}\vec{i}' + \frac{dy}{dt}\vec{j}' + \frac{dz}{dt}\vec{k}'} + x\frac{d\vec{i}'}{dt} + y\frac{d\vec{j}'}{dt} + z\frac{d\vec{k}'}{dt} =$$

$$= \frac{d'\vec{r}}{dt} + x(\vec{\omega} \times \vec{i}') + y(\vec{\omega} \times \vec{j}') + z(\vec{\omega} \times \vec{k}') =$$

$$= \frac{d'\vec{r}}{dt} + \vec{\omega} \times (x\vec{i}' + y\vec{j}' + z\vec{k}') = \frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r}$$

$$\boxed{\frac{d\vec{r}}{dt} = \frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r}} \xrightarrow{\text{Calquera vector } \vec{A}} \frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\omega} \times \vec{A}$$

$$\vec{v}_I = \vec{v} + \vec{\omega} \times \vec{r}$$

velocidade medida no sistema inercial — velocidade medida no sistema non inercial — velocidade de arrastre

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d'\vec{r}}{dt} + \vec{\omega}\times\vec{r}\right) = \frac{d}{dt}\left(\frac{dx}{dt}\vec{i}\,' + \frac{dy}{dt}\vec{j}\,' + \frac{dz}{dt}\vec{k}\,'\right) + \frac{d}{dt}(\vec{\omega}\times\vec{r}) =$$

$$= \overbrace{\frac{d^2x}{dt^2}\vec{i}\,' + \frac{d^2y}{dt^2}\vec{j}\,' + \frac{d^2z}{dt^2}\vec{k}\,'}^{\text{I}} + \overbrace{\frac{dx}{dt}\cdot\frac{d\vec{i}\,'}{dt} + \frac{dy}{dt}\cdot\frac{d\vec{j}\,'}{dt} + \frac{dz}{dt}\cdot\frac{d\vec{k}\,'}{dt}}^{\text{II}} +$$

$$+ \underbrace{\frac{d\vec{\omega}}{dt}\times\vec{r}}_{\text{III}} + \underbrace{\vec{\omega}\times\frac{d\vec{r}}{dt}}_{\text{IV}}$$

I $\quad \frac{d^2x}{dt^2}\vec{i}\,' + \frac{d^2y}{dt^2}\vec{j}\,' + \frac{d^2z}{dt^2}\vec{k}\,' = \frac{d'^2\vec{r}}{dt^2} = \vec{a}\,' \rightarrow$ Aceleración no Sistema non Inercial.

II $\quad \frac{dx}{dt}\cdot\frac{d\vec{i}\,'}{dt} + \frac{dy}{dt}\cdot\frac{d\vec{j}\,'}{dt} + \frac{dz}{dt}\cdot\frac{d\vec{k}\,'}{dt} = \frac{dx}{dt}(\vec{\omega}\times\vec{i}\,') + \frac{dy}{dt}(\vec{\omega}\times\vec{j}\,') +$

$$+ \frac{dz}{dt}(\vec{\omega}\times\vec{k}\,') = \vec{\omega}\times\left(\frac{dx}{dt}\vec{i}\,' + \frac{dy}{dt}\vec{j}\,' + \frac{dz}{dt}\vec{k}\,'\right) =$$

$$= \vec{\omega}\times\frac{d'\vec{r}}{dt} = \vec{\omega}\times\vec{v}\,'$$

III $\quad \frac{d\vec{\omega}}{dt}\times\vec{r} = \underbrace{\vec{\alpha}}_{\text{aceleración angular}}\times\vec{r}$

IV $\quad \vec{\omega}\times\frac{d\vec{r}}{dt} = \vec{\omega}\times\left(\frac{d'\vec{r}}{dt} + \vec{\omega}\times\vec{r}\right) = \vec{\omega}\times\vec{v}\,' + \vec{\omega}\times(\vec{\omega}\times\vec{r})$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}\,' + \vec{\omega}\times\vec{v}\,' + \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v}\,' + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}\,' + 2\vec{\omega}\times\vec{v}\,' + \vec{\alpha}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\boxed{m\frac{d^2\vec{r}}{dt^2} = \vec{F} - m\frac{d^2\vec{R}}{dt^2}}$$

$$\boxed{m\overset{\frac{d'^2\vec{r}}{dt^2}}{\vec{a}\,'} = \vec{F} - m\Big[\underbrace{\frac{d^2\vec{R}}{dt^2}}_{\text{translacional}} + \underbrace{2\vec{\omega}\times\overset{\frac{d'\vec{r}}{dt}}{\vec{v}\,'}}_{\text{Coriolis}} + \underbrace{\vec{\alpha}\times\vec{r}}_{\text{Azimutal}} + \underbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{Centrífuga}}\Big]}$$

Lei de Newton para sistemas non inerciais

→ Forza Azimutal ou Forza de Euler:

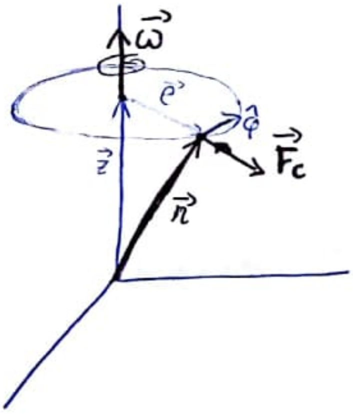
$$\boxed{\vec{F}_{azimutal} = m \cdot \vec{a}_{Azim} = -m\,\vec{\alpha} \times \vec{r}}$$

↳ Para que haxa forza azimutal:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} \neq 0 \Rightarrow \vec{\omega} \neq cte$$

→ Forza Centrifuga:

$$\boxed{\vec{F}_c = m \cdot \vec{a}_c = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})}$$



$$\vec{r} = \vec{e} + \vec{z}$$

$\vec{\omega} \parallel \vec{z}$

$$\vec{\omega} \times \vec{r} = \vec{\omega} \times (\vec{e} + \vec{z}) = \vec{\omega} \times \vec{e} + \cancel{\vec{\omega} \times \vec{z}}^{\,0} = (\omega \cdot e)\,\hat{\varphi}$$

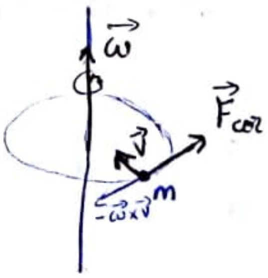
$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega\,\hat{z} \times (\omega \cdot e)\,\hat{\varphi} = -\omega^2 e\,\hat{e}$$

$$\vec{F}_c = m\omega^2 e\,\hat{e}\ , \quad \boxed{\vec{F}_c = m\omega^2 \vec{e}}$$

→ Forza de Coriolis:

$$\boxed{\vec{F}_{cor} = -2m\,\vec{\omega} \times \vec{v}}$$

$\vec{F}_{cor} = 0$ se $\vec{v} = 0$ ou $\vec{v} \parallel \vec{\omega}$



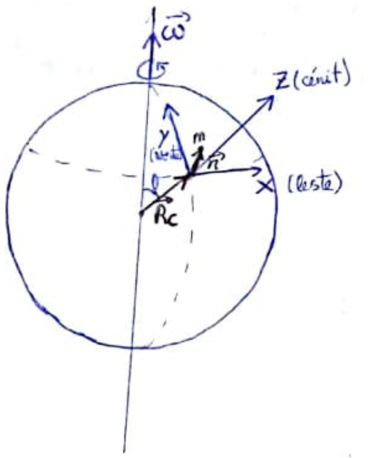
→ Forza translacional:

$$\boxed{\vec{F}_{trans} = -m\frac{d^2\vec{R}}{dt^2}}$$

forza fictícia (inercial) cando o sistema non rota

⊛ Sempre podemos elexir un S.R. no que a partícula se mova libremente: $m\frac{d'^2\vec{r}}{dt^2} = \vec{F} - m\frac{d^2\vec{R}}{dt^2}$

→ Movemento relativo para un observador na superficie da terra:



Lei de Newton para sistemas non inerciais:

$$m\vec{a} = \vec{F} - m\left[\frac{d^2\vec{R}}{dt^2} + 2\vec{\omega}\times\vec{v} + \vec{\alpha}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right]$$

$$\vec{\alpha} = 0 \Rightarrow \vec{\alpha}\times\vec{r} = 0$$
$(\vec{\omega} = cte)$

$$\vec{r} + \vec{R}_c \simeq \vec{R}_c \quad \left[|\vec{R}_c| >>> |\vec{r}|\right]$$

$$\frac{d^2\vec{R}_c}{dt^2}\,? \quad \frac{d\vec{R}_c}{dt} = \cancel{\frac{d'\vec{R}_c}{dt}}^{\,0} + \vec{\omega}\times\vec{R}_c$$

$$\vec{F} = \vec{F}' + m\vec{g} \qquad \frac{d^2\vec{R}_c}{dt^2} = \frac{d(\vec{\omega}\times\vec{R}_c)}{dt} = \vec{\omega}\times\frac{d\vec{R}_c}{dt} = \vec{\omega}\times(\vec{\omega}\times\vec{R}_c)$$

$$m\vec{a} = \vec{F}' + m\vec{g} - m\left[\vec{\omega}\times(\vec{\omega}\times\vec{R}_c) + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\vec{v}\right] =$$

$$= \vec{F}' + m\vec{g} - m\left[\vec{\omega}\times(\vec{\omega}\times\underbrace{(\vec{R}_c+\vec{r})}_{\simeq\,\vec{R}_c}) + 2\vec{\omega}\times\vec{v}\right]$$

$$|\vec{\omega}\times(\vec{\omega}\times\vec{R}_c)| = \omega^2\cdot R_c\cdot\sin\theta\cdot\sin\frac{\pi}{2} \leq \omega^2\cdot R_c \simeq 0'033\ m/s^2$$

$$\left.\begin{aligned} \omega &= \frac{2\pi}{86400\,s} \simeq 7'3\cdot10^{-5}\,s^{-1} \\ \omega^2 &\simeq 5'3\cdot10^{-9}\,s^{-2} \\ R_c &\simeq 6'371\cdot10^6\,m \end{aligned}\right\}$$

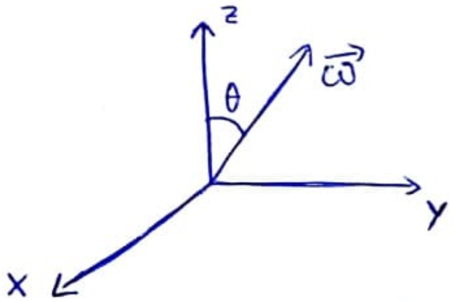
$|\vec{g}| \ggg 0'032 \ m/s^2$

Despreciamos $\vec{\omega} \times (\vec{\omega} \times (\vec{R}_c))$ frente a $\vec{g}$:

$$\boxed{m\vec{a} \approx \vec{F}' + m\vec{g} \underbrace{- 2m\vec{\omega} \times \vec{v}}_{\text{forza de coriolis}}}$$

Desviación dunha partícula ao lanzala :

$$\vec{F}_{coriolis} = -2m\vec{\omega} \times \vec{v}$$



$$\vec{\omega} = \omega \sin\theta \, \vec{j}' + \omega \cos\theta \, \vec{k}'$$
$$\vec{v} = v_x \vec{i}' + v_y \vec{j}' + v_z \vec{k}'$$

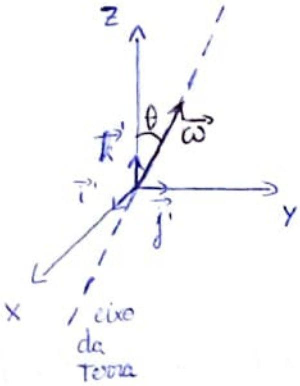
$$\vec{\omega} \times \vec{v} = \begin{vmatrix} \vec{i}' & \vec{j}' & \vec{k}' \\ 0 & \omega \sin\theta & \omega\cos\theta \\ v_x & v_y & v_z \end{vmatrix} =$$

$$= (v_z \omega \sin\theta - v_y \omega \cos\theta)\vec{i}' + v_x \omega \cos\theta \, \vec{j}' - v_x \omega \sin\theta \, \vec{k}'$$

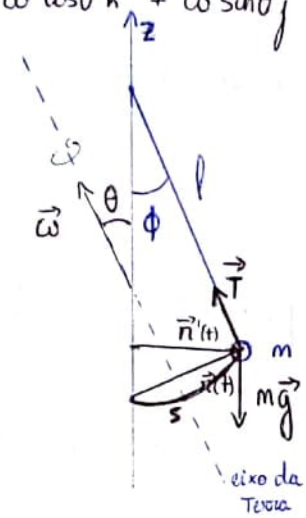
$$\boxed{\vec{F}_{coriolis} = 2m\omega \left[(v_y \cos\theta - v_z \sin\theta)\vec{i}' - v_x \cos\theta \, \vec{j}' + v_x \sin\theta \, \vec{k}' \right]}$$

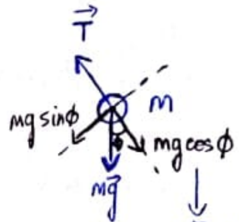
Dirección de $\vec{v}$	$\longrightarrow$ Dirección da forza de coriolis
Norte $v_y > 0$	Leste $(\vec{i}')$
Leste $v_x > 0$	Sur $(-\vec{j}')$ e arriba $(\vec{k}')$
Sur $v_y < 0$	Oeste $(-\vec{i}')$
Oeste $v_x < 0$	Norte $(\vec{j}')$ e abaixo $(-\vec{k}')$
Arriba $v_z > 0$	Oeste $(-\vec{i}')$
Abaixo $v_z < 0$	Leste $(\vec{i}')$

→ Péndulo de Foucault:



$$\vec{\omega} = \omega\cos\theta\,\vec{k}' + \omega\sin\theta\,\vec{j}'$$





esta compoñente do peso anúlase ca tensión

A outra compoñente é:

$$\vec{F}_G = \vec{T} + m\vec{g} = -mg\sin\phi\,\hat{r}$$

$$m\frac{d'^2\vec{r}}{dt^2} = \vec{T} + m\vec{g} - 2m\vec{\omega}\times\vec{v}$$

$$m\frac{d'^2\vec{r}}{dt^2} = \vec{F}_G - 2m\vec{\omega}\times\vec{v}$$

$$\phi = \frac{s}{l} \approx \frac{r}{l} \Longrightarrow \boxed{\sin\phi \approx \frac{r}{l}} \longrightarrow \vec{F}_G = -mg\sin\phi\,\hat{r} = -mg\frac{r}{l}\hat{r} = -\frac{mg}{l}\vec{r}$$

Supoñemos pequenas as oscilacións

$$\underbrace{\underbrace{\phi \approx 0}\quad \phi \approx \sin\phi}$$

Na práctica conséguese isto aumentando moito a lonxitude l

$$\boxed{\omega_0 = \sqrt{\frac{g}{l}}}$$

$$\vec{F}_G = -m\omega_0^2\vec{r}$$

$$m\frac{d'^2\vec{r}}{dt^2} = -m\omega_0^2\vec{r} - 2\vec{\omega}\times\vec{v}$$

$$\boxed{\frac{d'^2\vec{r}}{dt^2} = -\omega_0^2\,\vec{r} - 2\,\vec{\omega}\times\vec{v}}$$

Expresión que xa obtivemos para a forza de coriolis:

$$\vec{F}_{cor} = -2m\vec{\omega}\times\vec{v}$$

$$-2\,\vec{\omega}\times\vec{v} = 2\omega\left[(v_y\cos\theta - v_z\sin\theta)\,\vec{i}\,' - v_x\cos\theta\,\vec{j}\,' + v_x\sin\theta\,\vec{k}\,'\right]$$

Grazas á aproximación de que o ángulo é moi pequeno ($\phi \simeq 0$) podemos considerar que a partícula oscila no plano XY, de forma que $v_z \simeq 0$, Sendo a compoñente da forza de Coriolis no plano XY: $\vec{F}_{cor} = 2m\omega\left[v_y\cos\theta\,\vec{i}\,' - v_x\cos\theta\,\vec{j}\,'\right]$

$$\vec{a}_{cor} = -2\,\vec{\omega}\times\vec{v} = 2\omega\left[v_y\cos\theta\,\vec{i}\,' - v_x\cos\theta\,\vec{j}\,'\right]$$

Se consideramos: $\vec{r} = x\,\vec{i}\,' + y\,\vec{j}\,'$,

das compoñentes en x e y da Lei de Newton obtemos as seguintes ecuacións:

$$\left\{\begin{aligned} \ddot{x} &= -\omega_0^2\,x + 2\omega\cos\theta\,\dot{y} \\ \ddot{y} &= -\omega_0^2\,y - 2\omega\cos\theta\,\dot{x} \end{aligned}\right.$$

Ecuacións acopladas

$$\ddot{x} + \omega_0^2\,x = 2\omega\cos\theta\,\dot{y}$$

$\circledast$ $\omega\cos\theta = \omega_z$

$$+\ (\ddot{y} + \omega_0^2\,y = -2\omega\cos\theta\,\dot{x})\cdot i$$

$$(\ddot{x} + i\ddot{y}) + \omega_0^2\,(x+iy) = 2\omega_z\,\underbrace{(\dot{y} - i\dot{x})}_{-i(\dot{x}+i\dot{y})}$$

$$\boxed{q = x + iy}$$

$$\ddot{q} + \omega_0^2\,q = -2i\omega_z\,\dot{q}$$

$$\boxed{\ddot{q} + 2i\omega_z \dot{q} + \omega_0^2 q = 0}$$ → Ecuación diferencial de 2ª orde con coeficientes constantes homoxénea

$$q = C \cdot e^{\lambda t}$$

$$C \cdot \lambda^2 e^{\lambda t} + 2i\omega_z C \cdot \lambda \cdot e^{\lambda t} + \omega_0^2 \cdot e^{\lambda t} = 0$$

$$\boxed{\lambda^2 + 2i\omega_z \lambda + \omega_0^2 = 0}$$ → Ecuación característica

$$\lambda = \frac{-2i\omega_z \pm \sqrt{-4\omega_z^2 - 4\omega_0^2}}{2} = -i\omega_z \pm i\sqrt{\omega_z^2 + \omega_0^2}$$

$$\boxed{q = A \cdot e^{(-i\omega_z + i\sqrt{\omega_z^2+\omega_0^2})t} + B \cdot e^{(-i\omega_z - i\sqrt{\omega_z^2+\omega_0^2})t}}$$

Levamos a cabo unha nova aproximación:

$$\omega_z = \omega \cdot \cos\theta \leq \omega \simeq 7'3 \cdot 10^{-5} s^{-1}, \quad \omega^2 \simeq 5'3 \cdot 10^{-9} s^{-2}$$

$$\omega_0 = \sqrt{\frac{g}{l}} \simeq \sqrt{\frac{10\, m\, s^{-2}}{10\, m}} \simeq 1\, s^{-1}, \quad \omega_0^2 \simeq 1\, s^{-2}$$

$(l = 10m)$ p. e.

$$\boxed{\omega_z^2 <<<< \omega_0^2}$$

$$\boxed{\sqrt{\omega_z^2+\omega_0^2} \simeq \sqrt{\omega_0^2} = \omega_0}$$

desprezamos ω_z^2 fronte a ω_0^2

$$\boxed{q = e^{-i\omega_z t} \underbrace{\left(A \cdot e^{i\omega_0 t} + B \cdot e^{-i\omega_0 t}\right)}_{\bar{q}}}$$

$\boxed{\bar{q}}$ → solución para un péndulo simple sen coriolis

Euler: $\boxed{e^{i\alpha} = \cos\alpha + i \sin\alpha}$

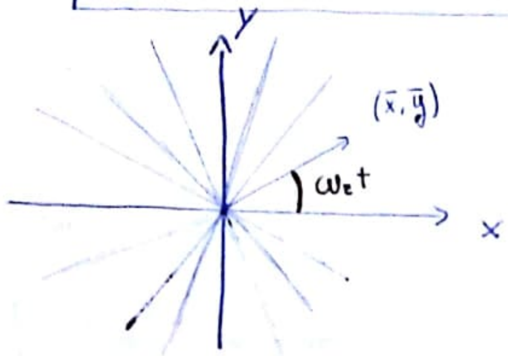
$$\bar{q} = \bar{x} + i\bar{y} = A\left[\cos(\omega_0 t) + i\sin(\omega_0 t)\right] + B\left[\cos(\omega_0 t) - i\sin(\omega_0 t)\right] =$$

$$= \underbrace{(A+B)}_{C_1}\cos(\omega_0 t) + \underbrace{i(A-B)}_{C_2}\sin(\omega_0 t)$$

$$\boxed{\bar{q} = \bar{x} + i\bar{y} \quad \begin{cases} \bar{x} = C_1 \cdot \cos(\omega_0 t) \\ \bar{y} = C_2 \cdot \sin(\omega_0 t) \end{cases}}$$

$$x + iy = \left(\cos(\omega_z t) - i\,\sin(\omega_z t)\right)\left(\bar{x} + i\bar{y}\right)$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\omega_z t & \sin\omega_z t \\ -\sin\omega_z t & \cos\omega_z t \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}}$$



$$\boxed{\underset{sdc}{\omega_z} = \omega_0 \cdot \cos\theta_{sdc} \simeq 0'17°/min}$$

Tema 1. Mecánica de Newton

1.1. a) Unha corda conecta dúas masas m_1 e m_2 penduradas dunha polea co soporte fixo (Fig. 1). Atopar as aceleracións de cada unha das masas.

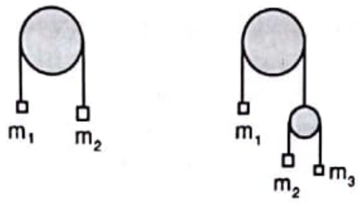
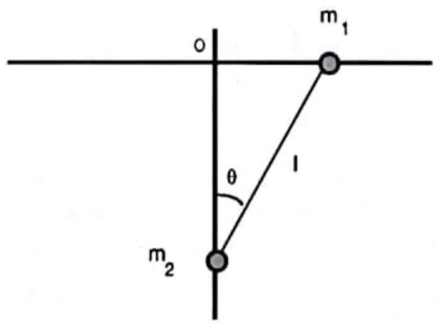


Fig. 1 Fig. 2

b) Considérese agora a polea dobre da Fig. 2 da que penduran m_1, m_2 e m_3. Atopar as aceleracións neste caso. Desprezar as masas das cordas e poleas nos dous apartados

1.2. Dúas bolas de igual masa m_1 e m_2 ($m_1 = m_2 = m$) conectadas entre si por unha corda inextensible de lonxitude L e masa desprezable móvense nun plano vertical de tal xeito que m_1 esvara sen rozamento ensartada nun arame horizontal mentras que m_2 faino nun arame vertical. Obter as ecuacións do movemento das masas, a tensión da corda e as forzas de reacción dos arames en función do ángulo θ que forma a corda coa horizontal, considerando que inicialmente m_2 está no punto (x=0,y=0) do plano XY e m_1 en (L,0) con velocidades nulas.



1.3. Un bloque de masa m esvara sen rozamento sobre un plano inclinado de ángulo θ. O plano inclinado atópase sobre unha mesa horizontal lisa. Calcular a aceleración do plano inclinado.

1.4. Unha copa de cristal moi fino ten forma semiesférica de radio $r = 5$

cm. O cristal pode resistir unha forza perpendicular de ata 2 N. Se se solta unha bola de chumbo de 100 g dentro da copa dende o borde superior e se permite que esvare por riba da superficie da copa, ¿en que punto se rompería o cristal? (Desprezar o radio da bola e a forza de rozamento).

1.5. Unha partícula de masa m móvese polo plano XY con vector de posición $\vec{r} = a\, cos(wt)\, \vec{i} + b\, sen(wt)\, \vec{j}$ onde a e b son constantes positivas sendo $a > b$. Demostrar a) que a traxectoria da partícula é unha elipse, b) que a forza vai dirixida á orixe de coordenadas, c) que o traballo realizado pola forza nunha volta completa é nulo, d) que a forza é conservativa.

1.6. En coordenadas cartesianas é trivial ver que unha partícula libre no plano se move en liña recta. Demostrar este mesmo resultado utilizando coordenadas polares.

1.7. Un cable flexible de lonxitude total l pendura unha porción l_0 do extremo dunha mesa. Desprezando o rozamento, calcular a porción de cable que pendura da mesa pasado un tempo t.

1.8. Unha gota de auga cae nunha atmósfera saturada de vapor de auga. Durante a caída o vapor vaise condensando na gota polo que aumenta o seu tamaño e masa. Calcular a masa, velocidade e posición da partícula en función do tempo. Supoñer que o aumento de masa por unidade de tempo é proporcional á superficie da gota. Considerar que a gota ten forma esférica, densidade uniforme e que inicialmente se atopa en repouso. Desprezar o rozamento.

1.9. Dúas esferas de ferro de densidade $\rho = 7500$ kg/m^3 e radios $r = 1$ e 5 cm déixanse caer dende unha altura de 15 m. Durante a caída están sometidas a unha forza de rozamento $F_{roz} = -C(rv)^2$, con $C = 1$ kg/m^3. **a)** Calcular a velocidade terminal v_t de cada bola. **b)** Determinar a velocidade de caida das bolas en función do tempo. **c)** Atopar unha expresión aproximada para o tempo que require a bola de radio r en acadar a velocidade $v = 0.99 v_t$ **d)** Que bola chega primeiro ao chan. Xustificalo.

1.10. Unha pelota lánzase verticalmente cara arriba con velocidade inicial v_0. A resistencia do aire é proporcional ao cadrado da velocidade. Demostrar que a velocidade da pelota cando volve ao punto de partida é $v = v_0 v_T / \sqrt{v_0^2 + v_T^2}$

√ **1.11.** Un centro de forzas atrae a unha partícula de acordo coa lei $F = -mk^2/x^3$, onde m é a masa e k unha constante. Demostrar que o tempo necesario para que a partícula chegue ao centro de forzas dende unha distancia d é d^2/k.

√ **1.12.** a) Unha colisión elástica entre dúas partículas é aquela na que a enerxía cinética total se conserva. Considérese o caso en que unha das partículas está inicialmente en repouso. Utilizando a conservación de momento, probar que as velocidades finais das partículas son perpendiculares. b) Unha colisión totalmente inelástica é aquela na que as partículas quedan *pegadas* despóis do choque. Atopar a velocidade final para un choque de dúas masas m_1 e m_2.

√ **1.13.** Un coche de masa $m_1 = 2000$ kg viaxa cara ao sur e choca no centro dun cruce cun camión de masa $m_2 = 6000$ kg que viaxaba cara o oeste. Os restos dos dous vehículos móvense xuntos despois do choque exactamente na dirección suroeste. É razoable pensar que o camión circulaba a unha velocidade $v_2 = 100$ km/h? Qué fracción da enerxía cinética transformouse noutro tipo de enerxía durante o choque se $v_1 = 180$ km/h? $\rightarrow \frac{5}{8}$ ($\frac{1}{8}$ según solucións)

√ **1.14.** Un proxectil de 10 g incrústrase nun bloque de 990 g que está suxeito a unha parede por un resorte de masa desprezable de constante $k = 10^5$ dinas/cm. Sabendo que o resorte se comprime ata un máximo de 10 cm, calcular: *a)* Enerxía potencial máxima do sistema. *b)* Velocidade do conxunto proxectil-bloque xusto despois do choque. *c)* Velocidade do proxectil xusto antes da colisión.

√ **1.15.** Discutir, usando a conservación de momento, o movemento dun foguete cuia masa vai diminuíndo segundo os motores van expulsando os productos da combustión do fuel.

√ **1.16.** Unha partícula de masa m está sometida a unha forza dada por $F(x) = -F_0 \sinh(ax)$ onde $a > 0$. É a forza conservativa?. Representar a enerxía potencial. Calcular a frecuencia das oscilacións pequenas se existe un punto de equilibrio estable. Estimar a distancia na que o movemento harmónico deixa de ser unha aproximación adecuada.

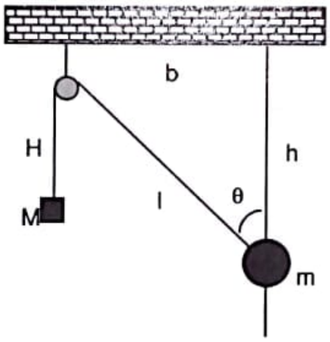
√ **1.17.** Unha partícula de masa m posee unha enerxía potencial $V(x) = C$ (C constante) se $|x| \leq L$ e $V(x) = 0$ no resto do espazo. A partícula, que inicialmente está na esquerda ($x \ll -L$) con velocidade v_0, móvese cara á dereita ($x \gg L$) "atravesando" a zona do potencial. **a)** Representar gráficamente a enerxía potencial e determinar o rango de enerxías que permita este movemento. **b)** Determinar a velocidade no extremo da dereita. **c)** Comparar o tempo que tarda a partícula en facer o percorrido de esquerda a dereita co caso en que $V(x) = 0$ en todo o espazo (partícula libre). Discutir o retraso acadado para C>0 e para C<0.

1.18. Escribir a enerxía total das masas da máquina de Atwood. Amosar que a ecuación do movemento obtense derivando a condición $E=$ constante (isto é certo para calquera sistema conservativo unidimensional).

1.19. No sistema da figura, a corda ten lonxitude l e a polea ten masa desprezable. Todo o movemento é no plano vertical baixo a acción da gravidade.

a) Escribir a enerxía potencial en función de θ e das constantes b, l.

b) Atopar as posicións de equilibrio e discutir para qué valores das masas existen e a súa estabilidade.



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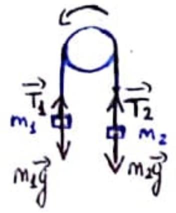
Tema 1: Mecánica de Neuton:

(1.1) a)



$(m_1 > m_2)$

Supoñendo a corda inextensible: $a_1 = a_2 = a$

2ª Lei Neuton:

(1) $m_1 g - T_1 = m_1 \cdot a$

(2) $T_2 - m_2 g = m_2 \cdot a$

$\vec{T_1} = \vec{T_2}$ $\hookrightarrow$ $T_2 = m_2 \cdot a + m_2 g = m_2 (a+g)$

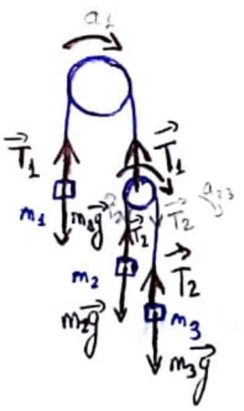
⊛ Desprezamos as masas das cordas e poleas

$m_1 g - m_2 (a+g) = m_1 \cdot a$

$m_1 g - m_2 g = m_1 a + m_2 a$

$$\boxed{a = \frac{(m_1 - m_2)}{(m_1 + m_2)} g}$$

b)



(1) $T_1 - m_1 g = m_1 \cdot a_1$

(p) $2T_2 - T_1 = 0$, $\boxed{T_1 = 2T_2}$

(2) $T_2 - m_2 g = m_2 (a_{23} - a_1)$

(3) $m_3 g - T_2 = m_3 (a_{23} + a_1)$

$$\boxed{T_2 = \frac{m_1 a_1 + m_1 g}{2}}$$

$a_{23} = \dfrac{(2m_2 + m_1) a_1 + (m_1 - 2m_2) g}{2 m_2}$ $\longleftarrow$ $m_1 a_1 + m_1 g - 2m_2 g = 2m_2 a_{23} - 2m_2 a_1$

$\boxed{a_{23}}$

$2m_3 g - m_1 a_1 - m_1 g = 2m_3 a_{23} + 2m_3 a_1$

$(2m_3 - m_1) g - (m_1 + 2m_3) a_1 = \dfrac{m_3 [(2m_2 + m_1) a_1 + (m_1 - 2m_2) g]}{m_2}$

$$\boxed{\begin{aligned} a_2 &= a_{23} - a_1 \\ a_3 &= a_{23} + a_1 \end{aligned}}$$

$m_2 (2m_3 - m_1) g - m_2 (m_1 + 2m_3) a_1 = m_3 (2m_2 + m_1) a_1 + m_3 (m_1 - 2m_2) g$

$g (2m_3 m_2 - m_2 m_1 - m_3 m_1 + 2m_3 m_2) = a_1 (2m_2 m_3 + m_1 m_3 + m_2 m_1 + 2m_2 m_3)$

$$\boxed{a_1} = \frac{4m_2 m_3 - m_1 (m_2 + m_3)}{4m_2 m_3 + m_1 (m_2 + m_3)} g = \frac{4 \frac{m_2 m_3}{m_2 + m_3} - m_1}{4 \frac{m_2 m_3}{m_2 + m_3} + m_1} g =$$

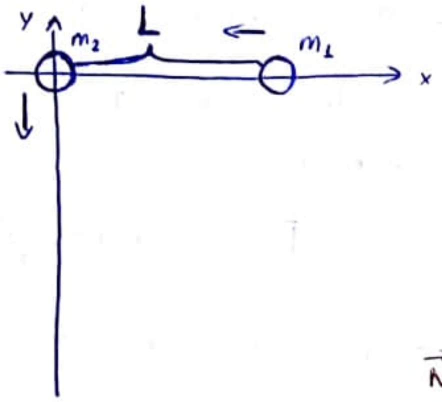
$$= \boxed{\frac{4 \mu_{23} - m_1}{4 \mu_{23} + m_1} g}$$

$\mu_{23} = \dfrac{m_2 m_3}{m_2 + m_3}$ masa reducida de m_2 y m_3

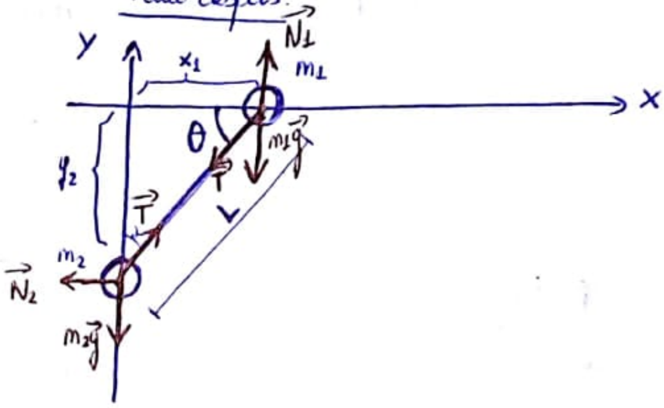
(1.2) $m_1 = m_2 = m$ Non hai rozamento.

Corda inextensible $L \longrightarrow$ masa desprezable $\vec{v}_{i_1} = \vec{v}_{i_2} = 0$

Inicialmente:



Pouco despois:



a) Ecuación do movemento en función da variable $\theta\ (\theta, \dot\theta, \ddot\theta)$:

Forzas: (2ª Lei Newton: $\sum_i \vec{F}_i = m\vec{a}$)

$$① \begin{cases} m_1 \cdot \ddot{x}_1 = -T\cos\theta \quad (1) \\ m_1 \cdot \ddot{y}_1 = N_1 - T\sin\theta - m_1 g = 0 \end{cases}$$

$$\ddot{y}_1 = 0 \quad (y_1 = cte) = 0$$

$$② \begin{cases} m_2\, \ddot{x}_2 = -N_2 + T\cos\theta = 0 \longrightarrow \ddot{x}_2 = 0 \quad (x_2 = cte) = 0 \\ m_2 \cdot \ddot{y}_2 = T\sin\theta - m_2 g \quad (2) \end{cases}$$

Ligaduras:

$$\begin{Bmatrix} x_1 = L\cos\theta \\ y_2 = -L\sin\theta \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dot{x}_1 = -L\sin\theta\,\dot\theta \\ \dot{y}_2 = -L\cos\theta\,\dot\theta \end{Bmatrix} \Rightarrow \begin{Bmatrix} \ddot{x}_1 = -L\left(\cos\theta\cdot\dot\theta^2 + \sin\theta\cdot\ddot\theta\right) \\ \ddot{y}_2 = -L\left(-\sin\theta\,\dot\theta^2 + \cos\theta\,\ddot\theta\right) \end{Bmatrix} (3)$$

→ Substituímos (3) en (1) e (2):

$$\begin{cases} \left(-m_1 L\left(\cos\theta\cdot\dot\theta^2 + \sin\theta\cdot\ddot\theta\right) = -T\cos\theta\right) \times \sin\theta\cdot m_2 \\ \left(-m_2 L\left(-\sin\theta\cdot\dot\theta^2 + \cos\theta\,\ddot\theta\right) = T\sin\theta - m_2 g\right) \times \cos\theta\cdot m_1 \end{cases} \oplus \longrightarrow$$

$$-m_1 m_2 L \sin\theta \left(\cos\theta\, \dot\theta^2 + \sin\theta \cdot \ddot\theta\right) = -T \sin\theta \cos\theta \cdot m_2$$

$$-m_2 m_1 L \cos\theta \left(-\sin\theta \cdot \dot\theta^2 + \cos\theta \cdot \ddot\theta\right) = T \sin\theta \cos\theta\, m_1 - m_2 m_1 g \cos\theta$$

$$-m_1 m_2 L \underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{1} \cdot \ddot\theta = \cancel{T \sin\theta \cos\theta \underbrace{(m_1 - m_2)}_{0}}^{\to 0} - m_1 m_2 g \cos\theta$$

$$-m_1 \cdot m_2 L\, \ddot\theta = -m_1 m_2 g \cos\theta$$

Ec. movemento: $\boxed{\ddot\theta = \frac{g}{L} \cos\theta}$ $\longrightarrow$ Ecuación dun péndulo simple: $\omega = \sqrt{g/L}$

⊛ Se usamos α (o ángulo da corda coa vertical) en vez de θ:

$$\theta = \frac{\pi}{2} - \alpha \quad \begin{bmatrix} \cos\theta = \sin\alpha \\ \ddot\theta = -\ddot\alpha \end{bmatrix} \Longrightarrow \boxed{\ddot\alpha = -\frac{g}{L} \sin\alpha}$$

⊛ Si $\alpha \ll 1$, $\alpha \sim 0 \Rightarrow \sin\alpha \approx \alpha$

$$\Downarrow \quad \ddot\alpha = -\frac{g}{L}\alpha$$

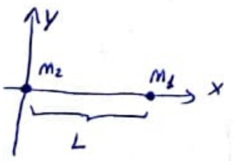
b) Tensión:

(1) $m_1 \ddot x_1 = -T\cos\theta$

$$T = \frac{-m_1 \ddot x_1}{\cos\theta} \overset{(3)}{=} \frac{-m_1\left(-L\left(\cos\theta \cdot \dot\theta^2 + \sin\theta \cdot \ddot\theta\right)\right)}{\cos\theta} = \frac{m_1 L \left(\cos\theta\, \dot\theta^2 + \sin\theta \cdot \ddot\theta\right)}{\cos\theta}$$

Usamos a conservación da enerxía para calcular $\dot\theta^2$:

Posición inicial



$$\left.\begin{array}{ll} x_1^0 = L & x_2^0 = 0 \\ y_1^0 = 0 & y_2^0 = 0 \\ v_1 = v_2 = 0 \Rightarrow T_1 = T_2 = 0 & \\ V = 0 \text{ en } y = 0 \Rightarrow V_1 = V_2 = 0 & \end{array}\right\} E_i = 0$$

A un ángulo θ:

$$E_f = T + V = \underbrace{\frac{1}{2} m_1 \dot x_1^2 + \frac{1}{2} m_2 \dot y_2^2}_{T} + \underbrace{\cancel{m_1 g y_1}^{0} + m_2 g\, y_2}_{V} =$$

$$\overset{(3)}{=} \frac{1}{2} m_1 \left(-L \sin\theta \cdot \dot{\theta}\right)^2 + \frac{1}{2} m_2 \left(-L \cos\theta \cdot \dot{\theta}\right)^2 - m_2 g \, L \sin\theta =$$

$$= \frac{1}{2} m_1 \, L^2 \sin^2\theta \, \dot{\theta}^2 + \frac{1}{2} m_2 \, L^2 \cos^2\theta \cdot \dot{\theta}^2 - m_2 g \, L \sin\theta =$$

$m = m_1 = m_2$

$$= \frac{1}{2} m \, L^2 \dot{\theta}^2 \underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{1} - m \cdot g \, L \sin\theta =$$

$$= \frac{1}{2} m \, L^2 \dot{\theta}^2 - m g \, L \sin\theta = 0 \quad (= E_i)$$

$$\boxed{\dot{\theta}^2 = \frac{2 g \sin\theta}{L}}$$

Derivamos para comprobar

$$2 \dot{\theta} \ddot{\theta} = 2 g \cos\theta \, \dot{\theta} \, \frac{1}{L}$$

$$\ddot{\theta} = \frac{g}{L} \cos\theta \quad \checkmark$$

$$T = \frac{m}{\cos\theta} \cancel{L} \left(\cos\theta \, \frac{2 g \sin\theta}{\cancel{L}} + \sin\theta \cdot \frac{g \cos\theta}{\cancel{L}}\right)$$

$$\boxed{T = 3 m g \sin\theta}$$

c) Normales:

① $N_1 - T\sin\theta - m_1 g = 0$, $N_1 = T\sin\theta + m_1 g$

$$N_1 = 3 m g \sin^2\theta + m g$$

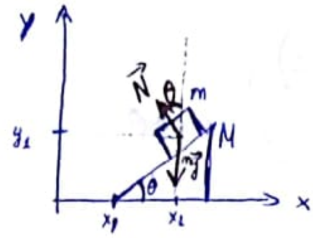
$$\boxed{N_1 = m g \left(1 + 3 \sin^2\theta\right)}$$

② $-N_2 + T\cos\theta = 0$, $N_2 = T\cos\theta$

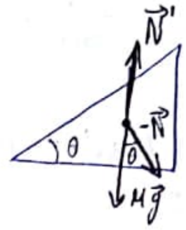
$$\boxed{N_2 = 3 m g \sin\theta \cos\theta}$$

$$\boxed{\circledast \ \text{Variables} = \text{Grados de libertad} - \text{Ligaduras} = \underbrace{2}_{\text{2 partículas en 1 dimensión}} - 1 = 1}$$

(1.3)



¿ a plano ?



Forzas sobre o bloque:

(1) $-N\sin\theta = m a_x$

(2) $N\cos\theta - mg = m a_y$

Forzas sobre o plano:

(3) $N\sin\theta = M a_x'$

(4) $N' - Mg - N\cos\theta = M a_y' = 0$

4 ecuacións, 5 incógnitas: N, N', a_x, a_y, a_x'

Ligadura: $\theta \to \tan\theta = \dfrac{y_1}{x_1 - x_p}$ $\quad (x_1 - x_p)\tan\theta = y_1$

derivando $\quad (\ddot{x}_1 - \ddot{x}_p)\tan\theta = \ddot{y}_1$

$$\boxed{\tan\theta = \frac{\ddot{y}_1}{\ddot{x}_1 - \ddot{x}_p}} \quad (5)$$

$\to \tan\theta = \dfrac{a_y}{a_x - a_x'}$

(1) + (3) : $N\sin\theta - N\sin\theta = m a_x + M a_x'$

$$m a_x = -M a_x' \quad (6)$$

(1) / (2) : $\dfrac{-N\sin\theta}{N\cos\theta} = \dfrac{m a_x}{m(a_y + g)}$, $\tan\theta = -\dfrac{a_x}{a_y + g}$ (7)

De (5), (6) e (7) sacamos a_x, a_y e a_x':

(7) $\to$ $a_y + g = \dfrac{-a_x}{\tan\theta} \Rightarrow a_y = -\dfrac{a_x}{\tan\theta} - g$

(5) $\to$ $a_y = \tan\theta\,(a_x - a_x')$

$-\dfrac{a_x}{\tan\theta} - g = \tan\theta\,(a_x - a_x')$

(6) $\to$ $a_x = -\dfrac{M}{m} a_x'$

$\dfrac{M}{m}\dfrac{a_x'}{\tan\theta} - g = \tan\theta\, a_x'\left(-\dfrac{M}{m} - 1\right)$

$$\times m\tan\theta \left(\frac{M}{m}\frac{a_x'}{\tan\theta} - g = \tan\theta\, a_x' \left(-\frac{M}{m} - 1\right) \right) \times m\tan\theta$$

$$M a_x' - mg\tan\theta = -a_x' \tan^2\theta\,(M+m)$$

$$\boxed{a_x' = \frac{m\,g\,\tan\theta}{M + (m+M)\tan^2\theta}}$$

Comprobación:

$\theta \to 0 \Rightarrow a_x' \longrightarrow 0$

$\theta \to \frac{\pi}{2} \Rightarrow a_x' \longrightarrow 0$

$m \to 0 \Rightarrow a_x' \longrightarrow 0$

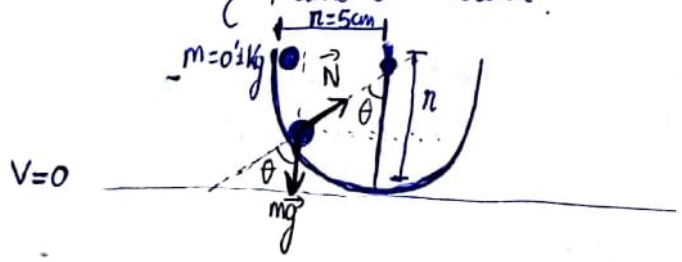
$m \gg M \Rightarrow a_x' \longrightarrow g/\tan\theta$

(1.4) Copa semiesférica: $r = 5\,cm$

O cristal aguanta unha forza ata $F = 2N$.

$m = 0'1\,Kg \longrightarrow$ esvara sen rozamento desde o borde superior

¿Punto de rotura?



Forzas na dirección normal á superficie:

$\sum F = m\cdot a$ (2ª Lei Newton)

$$N - mg\cos\theta = m\,a_n = m\,\frac{V^2}{r} \quad (1)$$

Non rozamento $\to$ Conservación da enerxía: $\boxed{E_i = Ef}$

$$\begin{cases} E_i = m\,g\,r \\ Ef = \frac{1}{2}mv^2 + mgh = \frac{1}{2}mv^2 + mg(r - r\cos\theta) \end{cases}$$

$$mgr = \frac{1}{2}mv^2 + mg(r - r\cos\theta)$$

$$0 = \frac{1}{2} v^2 - g\, r \cos\theta$$

$$\boxed{v^2 = 2 g r \cos\theta}$$

(1) $N = mg\cos\theta + m\frac{v^2}{r} = mg\cos\theta + m\frac{2gr\cos\theta}{r}$

$$\boxed{N = 3mg\cos\theta} \quad , \quad \cos\theta = \frac{N}{3mg}$$

A copa rompe cando $N = 2N$

$$\cos\theta = \frac{2N}{3\cdot(0'1Kg)\cdot(9'8m/s^2)} \quad , \quad \boxed{\theta \simeq 47'14^\circ}$$

O cristal rompe cando o ángulo formado pola vertical e o radio que une o centro e a bola de chumbo é de $47'14^\circ$.

(5) $\vec{r} = a\cos\omega t\,\vec{i} + b\sin\omega t\,\vec{j}$

a, b ctes , $a, b > 0$ $(a > b)$

a) Demostrar que a traxectoria é unha elipse:

$$\left.\begin{array}{l} x = a\cos\omega t \longrightarrow \cos\omega t = \frac{x}{a} \\ y = b\sin\omega t \longrightarrow \sin\omega t = \frac{y}{b} \end{array}\right\} \cos^2\omega t + \sin^2\omega t = 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Ecuación dunha elipse

b) Demostrar que a forza vai dirixida á orixe de coordenadas:

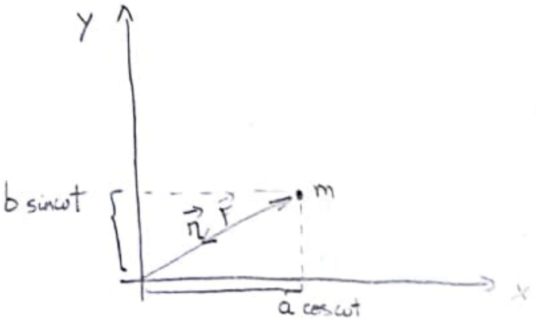
$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}$$

mesma dirección que a forza

$$\frac{d\vec{r}}{dt} = -a\omega \sin(\omega t)\,\hat{i} + \omega b \cos(\omega t)\,\hat{j} \qquad \vec{r} = a\cos(\omega t)\,\hat{i} + b\sin(\omega t)\,\hat{j}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\omega^2\cos(\omega t)\,\hat{i} - \omega^2 b\sin(\omega t)\,\hat{j} = -\omega^2\left(a\cos(\omega t)\,\hat{i} + b\sin(\omega t)\,\hat{j}\right)$$

$$= -\omega^2\,\vec{r}$$



$\vec{F} = -m\omega^2\vec{r} \rightarrow$ Proporcional a $\vec{r}$, pero de sentido contrario

$\vec{r}$ va del origen a la masa

$\vec{F}$ va de la masa hacia el origen

c) Demostrar que el trabajo en una vuelta completa es nulo:

$\omega t = 0 \rightarrow t = 0$

$\omega t = 2\pi \rightarrow t = \frac{2\pi}{\omega}$

$$\boxed{W} = \int \vec{F}\cdot d\vec{r} = \int -m\omega^2\vec{r}\cdot d\vec{r} =$$

$$= \int -m\omega^2\left(a\cos(\omega t)\,\hat{i} + b\sin(\omega t)\,\hat{j}\right)\cdot\left(-a\omega\sin(\omega t)\,\hat{i} + \omega b\cos(\omega t)\,\hat{j}\right)dt =$$

$$= -m\omega^2\int\left[-a^2\omega\sin(\omega t)\cos(\omega t) + b^2\omega\sin(\omega t)\cos(\omega t)\right]dt =$$

$$= -m\omega^2\int(b^2-a^2)\,\omega\left[\sin(2\omega t)\,dt\right] =$$

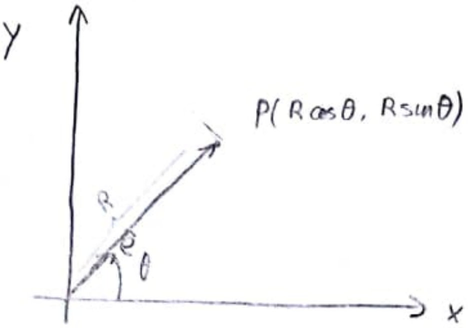
$$= m\omega^3(a^2-b^2)\int_0^{\frac{2\pi}{\omega}}\sin(2\omega t)\,dt = m\omega^3(a^2-b^2)\,\frac{1}{2\omega}\left(-\cos(2\omega t)\right)\Big]_0^{\frac{2\pi}{\omega}} =$$

$$= \frac{m\omega^2(a^2-b^2)}{2}\left(\cos 0 - \cos 4\pi\right) = \frac{m\omega^2(a^2-b^2)}{2}(1-1) = \boxed{0}$$

d) Demostrar que a força é conservativa $\vec{F} = -\omega^2 m x\,\hat{i} - \omega^2 m y\,\hat{j}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega^2 m x & -\omega^2 m y & 0 \end{vmatrix} = 0 \Rightarrow \vec{F} \text{ conservativa}$$

(6) Demostrar usando coordenadas polares que unha partícula libre no plano se move en liña recta:



$\vec{r} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j}$

$\hookrightarrow \boxed{\vec{r} = R\,\hat{e}}$

Cartesianas $\vec{r} = x\,\hat{i} + y\,\hat{j}$

$$\frac{d\vec{r}}{dt} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$$

$$\frac{d^2\vec{r}}{dt} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} \xrightarrow{2^a \text{ Ley Newton}} \vec{F} = m(\ddot{x}\,\hat{i} + \ddot{y}\,\hat{j}) =$$

$$= m\ddot{x}\,\hat{i} + m\ddot{y}\,\hat{j} \underset{\substack{\downarrow \\ \text{partícula libre}}}{=} 0$$

$$m\ddot{x} = 0, \quad m\ddot{y} = 0$$

$$\Downarrow$$

$$\ddot{x} = 0 \qquad \ddot{y} = 0$$

$$\Downarrow \qquad \Downarrow$$

$$\underbrace{\dot{x} = cte \qquad \dot{y} = cte}_{\text{movemento en liña recta}}$$

Polares planos $\vec{r} = R\,\hat{e}$

$$\boxed{\frac{d\vec{r}}{dt} = \dot{R}\,\hat{e} + R\cdot\dot{\hat{e}} = \vec{v}}$$

$\hat{e} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, $\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$$\dot{\hat{e}} = -\dot{\theta}\sin\theta\,\hat{i} + \dot{\theta}\cos\theta\,\hat{j} =$$

$$= \dot{\theta}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) = \dot{\theta}\cdot\hat{\theta}$$

$$\dot{\hat{\theta}} = -\dot{\theta}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = -\dot{\theta}\,\hat{e}$$

$$\boxed{\frac{d\vec{r}}{dt} = \dot{R}\,\hat{e} + R\dot{\theta}\,\hat{\theta}}$$

$$\boxed{\frac{d^2\vec{r}}{dt^2}} = \ddot{R}\,\hat{e} + \dot{R}\dot{\hat{e}} + (\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\theta} + (R\dot{\theta})\dot{\hat{\theta}} =$$

$$= (\ddot{R} - R\dot{\theta}^2)\hat{e} + (\dot{R}\dot{\theta} + \dot{R}\dot{\theta} + R\ddot{\theta})\hat{\theta} =$$

$$= (\ddot{R} - R\dot{\theta}^2)\hat{e} + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\theta} \boxed{=0}$$

partícula libre

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = 0$$

$$[1] \quad \ddot{R} - R\dot{\theta}^2 = 0 \,, \qquad \frac{\ddot{\theta}}{\dot{\theta}} = -2\frac{\dot{R}}{R} = \frac{\dot{\omega}}{\omega} \,, \quad -2\ln R = \ln\omega + c'$$

$$[2] \quad R\ddot{\theta} + 2\dot{R}\dot{\theta} = 0 \qquad \frac{d}{dt} \Rightarrow \cdot \,, \ \frac{d}{d\theta} \Rightarrow '$$

$$\ln R^{-2} = \ln\omega + \ln\tilde{C}$$

$$R^{-2} = C'\omega \,, \ \boxed{\omega = \frac{C}{R^2}}$$

$$\omega R^2 = C \Rightarrow m\omega R^2 = C$$

$$\underbrace{mR(\omega R) = L =}$$

momento angular, de porque $F_\theta = $

$R(\theta)$? $\rightarrow$ $(r = R)$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = r'\dot{\theta}$$

$$\ddot{r} = \underbrace{\dot{r}'}\dot{\theta} + r'\ddot{\theta} = r''\dot{\theta}^2 + \underbrace{r'\ddot{\theta}}_{[2]} = r''\dot{\theta}^2 - \frac{2r'\dot{r}\dot{\theta}}{r}$$

$$\dot{r}' = \frac{dr'}{dt} = \frac{dr'}{d\theta}\frac{d\theta}{dt} = r''\dot{\theta}$$

$$\ddot{r} = r''\dot{\theta}^2 - 2\frac{r'^2}{r}\dot{\theta}^2 = \left(r'' - 2\frac{r'^2}{r}\right)\dot{\theta}^2 \xrightarrow{[1]} \left(r'' - \frac{2r'^2}{r}\right)\dot{\theta}^2 = r\dot{\theta}^2 \,, \ \boxed{r'' - \frac{2r'^2}{r} - r = 0}$$

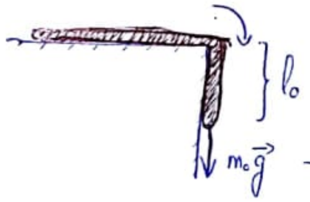
$$r = \frac{1}{u}$$

$$\boxed{1 = Ax + By} \leftarrow 1 = A\underbrace{r\cos\theta}_{x} + B\underbrace{r\sin\theta}_{y} \xleftarrow{u = \frac{1}{r}} u = A\cos\theta + B\sin\theta \leftarrow \boxed{u'' + u = 0}$$

$$\boxed{y = \frac{1}{B} - \frac{A}{B}x} \rightarrow \text{ecuación de una recta}$$

(7) Cable flexible de
- lonxitude total L
- masa M
- densidade $\rho = M/L$

$\circledast$ $F_{roz} = 0$



$\rightarrow \boxed{F(t) = m(t)\,g}$

¿ $l(t)$?

$$\Sigma F = \frac{dp}{dt} = \frac{d(M\cdot v)}{dt} = \cancelto{0}{\frac{dM}{dt}}\cdot v + \underbrace{M\cdot\frac{dv}{dt} = m(t)\,g}$$

$$\frac{dv}{dt} = \frac{dv}{dl}\cdot\frac{dl}{dt} = v\,\frac{dv}{dl}$$

$$m(t) = \rho\, l(t)$$

$$M v\frac{dv}{dl} = \rho\, l(t)\, g$$

$$M v\frac{dv}{dl} = \frac{M}{L}\, l(t)\, g$$

$$\int_0^v v'\,dv' = \frac{g}{L}\int_{l_0}^{l} l\, dl$$

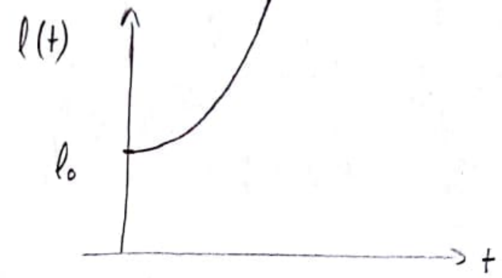
$$\left(\frac{v^2}{2}\right]_0^v = \frac{g}{L}\left(\frac{l^2}{2}\right]_{l_0}^{l}$$

$$\left\{\begin{aligned} v &= \sqrt{\frac{g}{L}(l^2 - l_0^2)} \\ v &= \frac{dl}{dt} \end{aligned}\right\} \leftarrow v^2 = \frac{g}{L}(l^2 - l_0^2)$$

$$\frac{d\ell}{dt} = \sqrt{\frac{g}{L}(\ell^2-\ell_0^2)} \quad , \quad \sqrt{\frac{g}{\ell}}\, dt = \frac{d\ell}{\sqrt{\ell^2-\ell_0^2}}$$

$$\sqrt{\frac{g}{L}}\int_0^t dt = \int_{\ell_0}^{\ell}\frac{d\ell}{\sqrt{\ell^2-\ell_0^2}} = \frac{1}{\ell_0}\int_{\ell_0}^{\ell}\frac{d\ell}{\sqrt{\left(\frac{\ell}{\ell_0}\right)^2-1}} = \left(\operatorname{arccosh}\frac{\ell}{\ell_0}\right]_{\ell_0}^{\ell} =$$

$$= \left(\operatorname{arccosh}\frac{\ell}{\ell_0} - \cancelto{0}{\operatorname{arccosh} 1}\right) = \operatorname{arccosh}\left(\frac{\ell}{\ell_0}\right)$$



$$\sqrt{\frac{g}{L}}\, t = \operatorname{arccosh}\left(\frac{\ell}{\ell_0}\right)$$

$$\boxed{\ell(t) = \ell_0 \cosh\left(\sqrt{g/L}\, t\right)}$$

⑧ Gota de auga ⟶ nunha atmósfera saturada de $\underbrace{\text{vapor de auga}}$.

Vaise condensando na gota

$\underbrace{\qquad\qquad}$ aumenta tamaño e masa

¿ $m(t)$, $v(t)$, $x(t)$?

⟶ O aumento de masa por unidade de tempo é proporcional á superficie da gota:

$$\frac{dm}{dt} = \alpha\, 4\pi R^2(t)$$

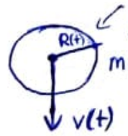
α ⟶ cte

$t = 0$

$R(0) = R_0$

$m(0) = m_0$

$\boxed{v(0) = 0}$ ⟵ Parte do repouso



↙ forma esférica

$\boxed{\rho \text{ uniforme}}$ (cte)

$$m(t) = \rho\,\frac{4}{3}\pi\, R^3(t)$$

$$\frac{dm}{dt} = \frac{4}{3}\pi\rho\; 3R^2(t)\cdot\frac{dR}{dt} = \rho\frac{dR}{dt}4\pi R^2(t)$$

$$\frac{dm}{dt} = \alpha\, 4\pi R^2(t) \longrightarrow \boxed{\alpha = \rho\frac{dR}{dt}} \qquad \frac{dR}{dt} = \frac{\alpha}{\rho} \equiv b$$

$$\int_{R_0}^{R} dR = b\int_0^t dt \longrightarrow \boxed{R(t) = R_0 + bt}$$

$$\boxed{m(t) = \rho \frac{4}{3} \pi (R_0 + bt)^3}$$

2ª Ley Newton :

$$\sum F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{dm}{dt} v + m \frac{dv}{dt}$$

$$\sum F = m(t)\, g$$

$$\frac{dp}{dt} = m(t)\, g \quad , \quad \int_0^{p(t)} dp = g \int_0^t m(t)\, dt$$

$$p(t) = m(t)\, v(t) = g\, \rho \frac{4}{3} \pi \int_0^t (R_0 + bt)^3\, dt =$$

$$= g\, \rho \frac{4}{3} \pi \left[\frac{(R_0 + bt)^4}{4b} \right]_0^t =$$

$$= g\, \rho \frac{4}{3} \pi \frac{1}{4b} \left[(R_0 + bt)^4 - R_0^4 \right]$$

$$v(t) = \frac{g \cancel{\rho} \cancel{\pi} \frac{1}{\cancel{3} b} \left[(R_0+bt)^4 - R_0^4 \right]}{\cancel{\rho} \frac{4}{\cancel{3}} \cancel{\pi} (R_0 + bt)^3} = \frac{g}{4b} \left[R_0 + bt - \frac{R_0^4}{(R_0+bt)^3} \right]$$

$$\boxed{v(t) = \frac{g}{4b} \left[R_0 + bt - \frac{R_0^4}{(R_0+bt)^3} \right]}$$

$$v(t) = \frac{g}{4} \left[\frac{R_0}{b} + t - \frac{R_0^4 / b}{(R_0 + bt)^3} \cdot \frac{1/b^3}{1/b^3} \right] = \frac{g}{4} \left[\frac{R_0}{b} + t - \frac{(R_0/b)^4}{\left(\frac{R_0}{b} + t\right)^3} \right]$$

$$\boxed{\circledast\ \beta \equiv \frac{R_0}{b}} \longrightarrow \boxed{v(t) = \frac{g}{4} \left[\beta + t - \frac{\beta^4}{(\beta + t)^3} \right]}$$

$$v(t) = \frac{dx(t)}{dt} ,$$

C.V. $\rightarrow$ $u = \beta + t$, $du = dt$; $t=0 \rightarrow u = \beta$; $t \rightarrow u = \beta + t$

$$\int_{x_0}^{x} dx = \int_0^t v(t)\, dt \ , \quad x - x_0 = \frac{g}{4} \int_0^t \left[\beta + t - \frac{\beta^4}{(\beta+t)^3} \right] dt = \frac{g}{4} \int_\beta^{\beta+t} \left[u - \frac{\beta^4}{u^3} \right] du =$$

$$= \frac{g}{4} \left(\frac{u^2}{2} + \beta^4 \frac{1}{2u^2} \right]_\beta^{\beta+t} = \frac{g}{8} \left((\beta+t)^2 + \beta^4 \frac{1}{(\beta+t)^2} - \beta^2 - \beta^4 \frac{1}{\beta^2} \right)$$

$$x - x_0 = \frac{g}{8}\left[(\beta+t)^2 - 2\beta^2 + \frac{\beta^4}{(\beta+t)^2}\right]$$

$$\boxed{x(t) = x_0 + \frac{g}{8}\left[(\beta+t)^2 - 2\beta^2 + \frac{\beta^4}{(\beta+t)^2}\right]}$$

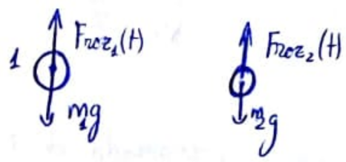
9

$$\left.\begin{array}{l} \rho = 7500\ Kg/m^3 \\ r_1 = 1\ cm \qquad r_2 = 5\ cm \end{array}\right\}\ \text{bolas}$$

$h = 15\,m \longrightarrow$ caída

$$\boxed{F_{roz} = -C(rv)^2}$$

$C = 1\ Kg/m^3$



a) ¿ Velocidade terminal de cada bola ?

$$m = V \cdot d = \frac{4}{3}\pi r^3 \rho \Rightarrow \left\{\begin{array}{l} m_1 = \frac{4}{3}\pi(1\cdot 10^{-2}m)^3 \cdot 7500\ Kg/m^3 = \frac{\pi}{100}\ Kg \\ m_2 = \frac{4}{3}\pi(5\cdot 10^{-2}m)^3 \cdot 7500\ Kg/m^3 = \frac{5}{4}\pi\ Kg \end{array}\right.$$

¿ $\dot{y}$? Cando $\ddot{y} = 0$

$$m\ddot{y} = mg - Cr^2\dot{y}^2 = 0$$

$$Cr^2\dot{y}^2 = mg, \qquad \dot{y} = \sqrt{\frac{mg}{Cr^2}}$$

$$\dot{y} = \sqrt{\frac{\frac{4}{3}\pi r^3 \rho g}{C r^2}} = \sqrt{\frac{4\pi r \rho g}{C\,3}}$$

$$v_{t_1} = \sqrt{\frac{4\pi \cdot 1\cdot 10^{-2}m \cdot 7500\ Kg/m^3 \cdot 9'8\ m/s^2}{3\cdot 1\ Kg/m^3}} = 55'49\ m/s$$

$$v_{t_2} = \sqrt{\frac{4\pi \cdot 5\cdot 10^{-2}m \cdot 7500\ Kg/m^3 \cdot 9'8\ m/s^2}{3\cdot 1\ Kg/m^3}} \simeq 124'07\ m/s$$

b) ¿ $v(t)$? $\quad m\frac{dv}{dt} = mg - Cr^2v^2 \quad$ ⊛ $t_0 = 0 \rightarrow v_0 = 0$

$$\int_0^v \frac{dv}{g - \frac{Cr^2}{m}v^2} = \int_0^t dt = t, \qquad \int_0^v \frac{dv}{g - \frac{Cr^2}{m}v^2} = \int_0^v \frac{1}{g}\frac{dv}{1 - \frac{Cr^2}{mg}v^2} =$$

$$= \frac{1}{g}\sqrt{\frac{mg}{Cr^2}} \int_0^v \frac{dv}{1-\left(\sqrt{\frac{Cr^2}{mg}}v\right)^2} = \frac{1}{g}\sqrt{\frac{mg}{Cr^2}} \; \operatorname{arctanh}\left(\sqrt{\frac{Cr^2}{mg}}\, v\right) = t$$

$$\sqrt{\frac{Cr^2}{mg}}\; v = \tanh\left(\sqrt{\frac{Cr^2 g^2}{mg}}\; t\right)$$

$$v(t) = \sqrt{\frac{4/3\,\pi r^3 \rho g}{C r^2}} \tanh\left(\sqrt{\frac{Cr^2 g}{4/3\,\pi r^3 \rho}}\; t\right)$$

$$\boxed{v(t) = \underbrace{\sqrt{\frac{4\pi r \rho g}{3C}}}_{v_t} \tanh\left(\sqrt{\frac{3Cg}{4\pi r \rho}}\; t\right)} = v_t \frac{1 - e^{-2\sqrt{\frac{3Cg}{4\pi r\rho}}\,t}}{1 + e^{-2\sqrt{\frac{3Cg}{4\pi r\rho}}\,t}}$$

c) Expresión aproximada do tempo da bola de radio r para acadar $v(t) = 0'99 v_t$.

$$v(t) = 0'99\, v_t \Rightarrow \tanh\left(\sqrt{\frac{3Cg}{4\pi r\rho}}\, t\right) = 0'99$$

$$t \simeq \operatorname{arctanh}(0'99) \cdot \sqrt{\frac{4\pi r\rho}{3Cg}} \qquad \sqrt{\frac{3Cg}{4\pi r\rho}} = b$$

$$\tanh(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}, \qquad v(t) = v_t \frac{1 - e^{-2bt}}{1 + e^{-2bt}}$$

$$0'99 = \frac{1 - e^{-2bt}}{1 + e^{-2bt}}, \qquad (1 + e^{-2bt})\,0'99 = 1 - e^{-2bt}$$

$$0'99 + 0'99\, e^{-2bt} = 1 - e^{-2bt}$$

$$1'99\, e^{-2bt} = 0'01$$

$$e^{-2bt} = \frac{0'01}{1'99}, \quad -2bt = \ln\left(\frac{0'01}{1'99}\right)$$

$$\boxed{t = \frac{1}{2}\sqrt{\frac{4\pi r\rho}{3Cg}}\,\ln(199)}$$

d) ¿Que bola chega primeiro ao chan?

$$m\frac{dv}{dt} = mg - Cr^2v^2 \qquad \frac{Cr^2}{m} = \frac{Cr^2}{\frac{4}{3}\pi r^3\rho} = \frac{3C}{4\pi\rho}\frac{1}{r}$$ — Maior na bola de menor radio

$$\frac{dv}{dt} = g - \frac{Cr^2v^2}{m} \qquad \frac{Cr_1^2}{m_1} > \frac{Cr_2^2}{m_2} \Rightarrow \underbrace{\left(\frac{dv}{dt}\right)_2 > \left(\frac{dv}{dt}\right)_1}$$

⇓

Chegará primeiro a bola de maior radio (r_2)

1.10

$\uparrow v_0 \qquad F_{roz} = \alpha \cdot v^2 \qquad y_0 = 0$

Demostrar que a velocidade cando a pelota volve a x_0 é

$v = v_0 v_T / \sqrt{v_0^2 + v_T^2}$

Cando sube: $\sum F = -mg - \alpha v^2 = \frac{dp}{dt} = m\frac{dv}{dt}$

$$m\frac{dv}{dt} = -mg - \alpha v^2$$

$$m \frac{dv}{mg + \alpha v^2} = -dt$$

$$\frac{1}{g}\int_{v_0}^{v} \frac{dv}{1 + \frac{\alpha}{mg} v^2} = -\int_0^t dt$$

$$\frac{1}{g}\sqrt{\frac{mg}{\alpha}}\left(\arctan\sqrt{\frac{\alpha}{mg}}\,v\right]_{v_0}^{v} = -t$$

. . .

Máis fácil

$$m\frac{dv}{dy}\cdot\frac{dy}{dt} = -mg - \alpha v^2$$

$$m\frac{dv}{dy}v = -mg - \alpha v^2$$

$$v\frac{dv}{dy} = -g - \frac{\alpha}{m}v^2$$

$$\int_{v_0}^{v}\frac{v\,dv}{g + \frac{\alpha}{m}v^2} = -\int_0^y dy$$

$$\frac{m}{2\alpha}\left(\log\left(g + \frac{\alpha}{m}v^2\right)\right]_{v_0}^{v} = -y$$

$$\boxed{y(v) = -\frac{m}{2\alpha}\log\left(\frac{g + \frac{\alpha}{m}v^2}{g + \frac{\alpha}{m}v_0^2}\right)}$$

$\boxed{h = y(0) =}$

$= -\frac{m}{2\alpha}\log\frac{g}{g + \frac{\alpha}{m}v_0^2} =$

$\boxed{= \frac{m}{2\alpha}\log\left(1 + \frac{\alpha}{mg}v_0^2\right)}$

Cando baixa: $\quad v_0 = 0 \quad , \quad y_0 = h \qquad (v \leq 0)$

$$m\frac{dv}{dt} = -mg + \alpha v^2$$

$$\frac{dv}{dt} = -g + \frac{\alpha}{m}v^2$$

$$\frac{dv}{dy}\,\frac{dy}{dt} = -g + \frac{\alpha}{m}v^2$$

$$\int_0^v \frac{dv\,v}{-g + \frac{\alpha}{m}v^2} = \int_h^y dy'$$

$$\frac{m}{2\alpha}\left(\log\left(-g + \frac{\alpha}{m}v^2\right)\right]_0^v = y - h$$

$$y(v) = h + \frac{m}{2\alpha}\left[\log\left(-g + \frac{\alpha}{m}v^2\right) - \log(-g)\right]$$

$$y(v) = h + \frac{m}{2\alpha}\log\left(\frac{-g + \frac{\alpha}{m}v^2}{-g}\right)$$

$$\boxed{y(v) = h + \frac{m}{2\alpha}\log\left(1 - \frac{\alpha}{mg}v^2\right)}$$

$$m\frac{dv}{dt} = -mg + \alpha v^2$$

$$v_T \Longleftrightarrow \frac{dv}{dt} = 0 \qquad 0 = -mg + \alpha v_T^2$$

$$\boxed{v_T^2 = \frac{mg}{\alpha}}$$

$$y(v) = \frac{m}{2\alpha}\log\left(1 + \frac{\alpha}{mg}v_0^2\right) + \frac{m}{2\alpha}\cdot\log\left(1 - \frac{\alpha}{mg}v^2\right) =$$

$$= \frac{m}{2\alpha}\log\left(1 + \frac{\alpha}{mg}v_0^2 - \frac{\alpha}{mg}v^2 - \frac{\alpha^2 v_0^2 v^2}{m^2g^2}\right) =$$

$$= \frac{m}{2\alpha}\log\left(1 + \frac{\alpha}{mg}(v_0^2 - v^2) - \frac{\alpha^2}{m^2g^2}v_0^2v^2\right) =$$

$$= \frac{m}{2\alpha}\log\left(1 + \frac{v_0^2 - v^2}{v_T^2} - \frac{v_0^2v^2}{v_T^4}\right)$$

$y = 0$ Cando volve ao chan

$$\log\Big(\underbrace{1 + \frac{v_0^2 - v^2}{v_T^2} - \frac{v_0^2v^2}{v_T^4}}_{=1}\Big) = 0$$

$$\frac{v_0^2 - v^2}{v_T^2} = \frac{v_0^2v^2}{v_T^4}$$

$$v_T^2(v_0^2 - v^2) = v_0^2v^2$$

$$v_T^2v_0^2 = v_0^2v^2 + v^2v_T^2$$

$$v^2 = \frac{v_T^2\cdot v_0^2}{v_T^2 + v_0^2} \Longrightarrow \boxed{v = \frac{v_T\, v_0}{\sqrt{v_0^2 + v_T^2}}} \quad q.e.d.$$

(11)

$$F = -m\frac{k^2}{x^3} \qquad \begin{matrix} m & \text{masa} \\ k & \text{cte} \end{matrix}$$

$t=0 \longrightarrow x = d \rightarrow$ reposo

$x = 0 \xrightarrow{\text{Demostrar}} t = \frac{d^2}{k}$

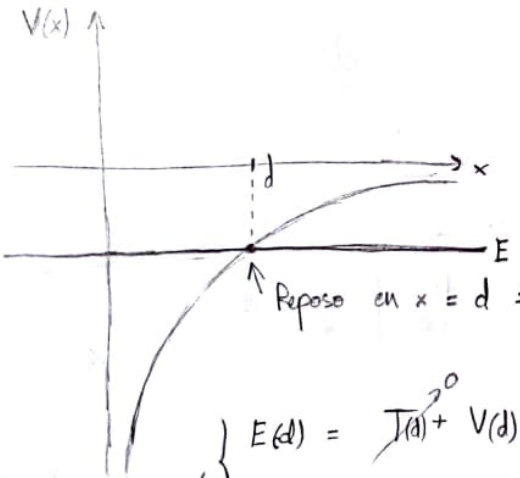
¿?

a) Por enerxía:

$F(x)$ en 1D $\longrightarrow$ conservativa : $\exists\, V(x) \,/\, F = -\frac{dV}{dx}$

$$\int dV = -\int F(x)\,dx$$

$$V = \int mk^2 \frac{1}{x^3}\,dx = -\frac{mk^2}{2}\frac{1}{x^2} + C^{=0} \begin{cases} x \to \infty \\ V(x) \to 0 \end{cases}$$



Reposo en $x = d \Rightarrow V = 0$, $T = 0$, $\boxed{E = V(d)}$

$$E(d) = \cancel{T(d)}^{0} + V(d) = -\frac{1}{2}mk^2\frac{1}{d^2}$$

$$E(x) = T(x) + V(x) = \frac{1}{2}mv(x)^2 - \frac{1}{2}mk^2\frac{1}{x^2}$$

$E(d) = E(x)$ (conservación enerxía)

$$-\frac{1}{2}mk^2\frac{1}{d^2} = \frac{1}{2}mv^2(x) - \frac{1}{2}mk^2\frac{1}{x^2}$$

$$-k^2/d^2 = v^2(x) - k^2/x^2$$

$$v^2(x) = k^2\left(\frac{1}{x^2} - \frac{1}{d^2}\right)$$

$$v(x) = \pm k\sqrt{\frac{1}{x^2} - \frac{1}{d^2}}$$

$\rightarrow v < 0$ (porque se mueve hacia la izquierda)

$$v = \frac{dx}{dt} = -k\sqrt{\frac{1}{x^2} - \frac{1}{d^2}}$$

$$\int_d^0 \frac{dx}{\sqrt{\frac{1}{x^2} - \frac{1}{d^2}}} = -k\int_0^{t_d} dt = -k t_d$$

$$\int_d^0 \frac{dx}{\sqrt{\frac{1}{x^2} - \frac{1}{d^2}}} = \int_d^0 \frac{x\,dx}{\sqrt{\frac{x^2}{x^2} - \frac{x^2}{d^2}}} = \int_d^0 \frac{x\,dx}{\sqrt{1 - \frac{x^2}{d^2}}} =$$

$$= -d^2 \int_d^0 \frac{\frac{-2x}{d^2}\,dx}{2\sqrt{1 - \frac{x^2}{d^2}}} = -d^2 \left(\sqrt{1 - \frac{x^2}{d^2}}\right]_d^0 =$$

$$= -d^2\left(\sqrt{1-0} - \sqrt{1 - \frac{d^2}{d^2}}\right) = -d^2$$

$$-k t_d = -d^2 \,, \quad \boxed{t_d = d^2/k} \quad \text{q.e.d.}$$

b) Usando a 2^a Lei de Newton

$$F(x) = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m v \frac{dv}{dx}$$

$$m v \frac{dv}{dx} = -m\frac{k^2}{x^3}$$

$$\int_0^v v\,dv = -k^2 \int_d^x \frac{dx}{x^3}$$

$$\frac{v^2}{\cancel{2}} = -k^2\left(\frac{1}{-\cancel{2}x^2}\right]_d^x \qquad v^2 = k^2\left(\frac{1}{x^2} - \frac{1}{d^2}\right)$$

$$\boxed{v(x) = \pm k\sqrt{\frac{1}{x^2} - \frac{1}{d^2}}}$$

(A partir de aqui xa é igual)

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1.12

a) Colisión elástica $\longrightarrow$ Consérvase a T total

$$v_{A_0} = 0 \quad m_A \xrightarrow{\quad\quad} v_{Af}$$
Choque
$$v_{B_0} \neq 0 \quad m_B \xrightarrow{\quad\quad} v_{Bf}$$

Usar a conservación do momento para demostrar que as velocidades finais das partículas son perpendiculares:

m_B O $\xrightarrow{v_{B_0}}$ O m_A

→ Conservación de T:
$$\cancel{T_{A_0}}^{0} + T_{B_0} = T_{Af} + T_{B_0}$$
$$\frac{\cancel{1}}{\cancel{2}} m_B v_{B_0}^2 = \frac{\cancel{1}}{\cancel{2}} m_A v_{Af}^2 + \frac{\cancel{1}}{\cancel{2}} m_B v_{Bf}^2$$
$$v_{B_0}^2 = \frac{m_A}{m_B} v_{Af}^2 + v_{Bf}^2$$

→ Conservación de $\vec{p}$

$$m_A \cancel{\vec{v}_{A_0}}^{0} + m_B \cdot \vec{v}_{B_0} = m_A \cdot \vec{v}_{Af} + m_B \vec{v}_{Bf}$$

Elevamos ao cadrado

$$(m_B \vec{v}_{B_0})^2 = (m_A \vec{v}_{Af} + m_B \vec{v}_{Bf})^2$$
$$m_B^2 \vec{v}_{B_0} \cdot \vec{v}_{B_0} = m_A^2 \vec{v}_{Af} \cdot \vec{v}_{Af} + m_B^2 \vec{v}_{Bf} \cdot \vec{v}_{Bf} + 2 m_A m_B \vec{v}_{Af} \vec{v}_{Bf}$$
$$m_B^2 v_{B_0}^2 = m_A^2 v_{Af}^2 + m_B^2 \cdot v_{Bf}^2 + 2 m_A m_B \vec{v}_{Af} \cdot \vec{v}_{Bf}$$
$$v_{B_0}^2 = \left(\frac{m_A}{m_B}\right)^2 v_{Af}^2 + v_{Bf}^2 + 2 \frac{m_A}{m_B} \vec{v}_{Af} \cdot \vec{v}_{Bf}$$

$$\frac{m_A}{m_B} v_{Af}^2 + \cancel{v_{Bf}^2} = \left(\frac{m_A}{m_B}\right)^2 v_{Af}^2 + \cancel{v_{Bf}^2} + 2 \frac{m_A}{m_B} \vec{v}_{Af} \cdot \vec{v}_{Bf}$$

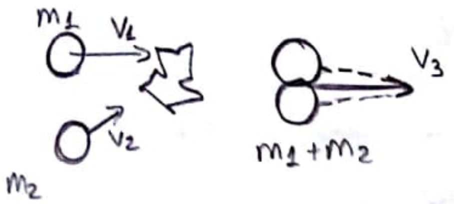
$$v_{Af}^2 = \frac{m_A}{m_B} v_{Af}^2 + 2 \vec{v}_{Af} \cdot \vec{v}_{Bf}$$
$$\boxed{2 \vec{v}_{Af} \cdot \vec{v}_{Bf} = \left(1 - \frac{m_A}{m_B}\right) v_{Af}^2}$$

$$\vec{v}_{Af} \cdot \vec{v}_{Bf} = 0 \iff 1 - \frac{m_A}{m_B} = 0, \; m_A = m_B$$

$$\boxed{\vec{v}_{Af} \perp \vec{v}_{Bf} \iff m_A = m_B}$$

b) Colisión totalmente inelástica $\equiv$ as partículas quedan pegadas despois do choque

¿ vf ? Choque de m_1 e m_2



$$\vec{p}_i = \vec{p}_f$$

$$\vec{p}_1 + \vec{p}_2 = \vec{P}_3$$

$$m_1 \cdot \vec{v}_1 + m_2 \cdot \vec{v}_2 = (m_1 + m_2)\,\vec{v}_3$$

$$\boxed{\vec{v}_3 = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}}$$

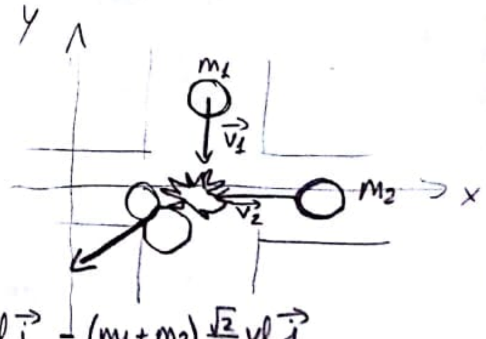
⊛ Se $\vec{v}_1 = 0$, $\vec{v}_3 = \dfrac{m_2 \vec{v}_2}{m_1 + m_2}$

(1.13) $m_1 = 2000\ Kg \longrightarrow$ viaxa cara ao sur $\vec{v}_1 = -v_1 \vec{j}$

$m_2 = 6000\ Kg \longrightarrow$ viaxa cara ao oeste $\vec{v}_2 = -v_2 \vec{i}$

Colisión totalmente inelástica $\longrightarrow$ $\vec{v}_f = -\frac{\sqrt{2}}{2} v_f \vec{i} - \frac{\sqrt{2}}{2} v_f \vec{j}$

É razoable pensar que $v_2 = 100\ Km/h$



$\boxed{\vec{p}_i = \vec{p}_f} \longrightarrow \vec{p}_1 + \vec{p}_2 = \vec{P}_3$

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = (m_1 + m_2)\,\vec{v}_f$$

$$\underbrace{-m_1 v_1 \vec{j} - m_2 v_2 \vec{i} = -(m_1+m_2)\frac{\sqrt{2}}{2} v_f \vec{i} - (m_1+m_2)\frac{\sqrt{2}}{2} v_f \vec{j}}_{\text{Igualando compoñentes:}}$$

$$\left.\begin{aligned} m_1 v_1 &= (m_1 + m_2)\frac{\sqrt{2}}{2} v_f \\ m_2 v_2 &= (m_1 + m_2)\frac{\sqrt{2}}{2} v_f \end{aligned}\right\} \quad \boxed{m_1 v_1 = m_2 v_2}$$

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Se o camión fose a $v_2 = 100\,Km/h$, o coche tería que ir a:

$$v_1 = \frac{m_2 \; v_2}{m_1} \;,\quad v_1 = \frac{6000\,Kg \cdot 100\,Km/h}{2000\,Kg} = \boxed{300\,Km/h}$$

Non parece moi razoable

Se $v_1 = 180\,Km/h$, ¿ Que fracción de T se transformou noutro tipo de enerxía ?

$$\frac{T_{despois}}{T_{antes}} = \frac{\frac{1}{2}(m_1+m_2)\,v_f^2}{\frac{1}{2}m_1 v_1^2 + \frac{1}{2} m_2 v_2^2} = \frac{(m_1+m_2)\,v_f^2}{m_1 v_1^2 + m_2 v_2^2}$$

$$\boxed{v_2 = \frac{m_1 \; v_1}{m_2}} \;,\quad v_2 = \frac{2000\,Kg \cdot 180\,Km/h}{6000\,Kg} = 60\,Km/h$$

$$\boxed{v_f = \frac{2}{\sqrt{2}} \; \frac{m_1 v_1}{(m_1+m_2)}} \qquad 180\,Km/h = 50\,m/s$$

$$\frac{T_{despois}}{T_{antes}} = \frac{\cancel{(m_1+m_2)} \; \frac{m_1^2 \, v_1^2}{(m_1+m_2)^{\cancel{2}}} \cdot 2}{m_1 \, v_1^2 + m_2 \; v_2^2} = \frac{2 \; m_1^{\cancel{2}} \;\; v_1^{\cancel{2}}}{(m_1+m_2)\left(\cancel{m_1} v_1^{\cancel{2}} + \cancel{m_2}\left(\frac{v_1 \, m_1}{m_2}\right)^{\cancel{2}}\right)} =$$

$$= \frac{2 \; m_1 \; v_1}{(m_1+m_2)\,(v_1 + 1)}$$

$$\frac{T_{despois}}{T_{antes}} = \frac{2 \cdot 2000\,Kg \cdot 50\,m/s}{8000\,Kg \;\left(50\,m/s + 50\,m/s \; \frac{1}{3}\right)} = \boxed{\frac{3}{8}}$$

$$1 - \frac{3}{8} = \frac{5}{8} \longrightarrow$$

$\boxed{\frac{5}{8} \text{ da } T \text{ inicial convertéronse noutro tipo de enerxía}}$

1.14

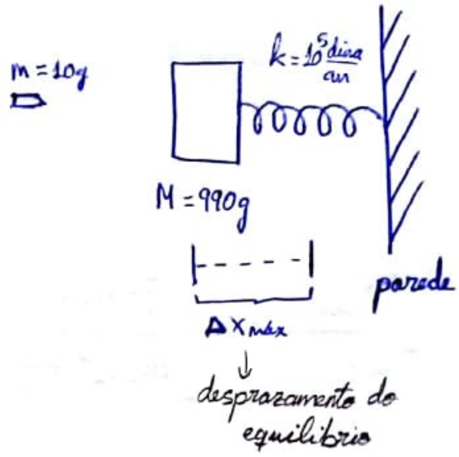
$m_p = 10\,g \rightarrow$ proxectil $\xrightarrow{\text{incrústase}}$

$M_B = 990\,g \rightarrow$ Bloque suxeito á parede por un resorte

$k = 10^5$ dinas/cm

Comprímese ata un máximo de 10 cm

a) ¿ Ep máx do sistema ?



$$\boxed{F(x) = -kx} \rightarrow \text{resorte}$$

$$F(x) = -\frac{dV(x)}{dx}$$

$$V = -\int F(x)\,dx + C_1 =$$

$$= k\frac{x^2}{2} + C_1 \longrightarrow C_1 = 0 \quad \boxed{V(0) = 0}$$

10 cm = 0'1 m

$$k = 10^5\frac{\text{dinas}}{\text{cm}} \cdot \frac{10^{-5}N}{1\,\text{dina}} \cdot \frac{1\,\text{cm}}{10^{-2}m} = \boxed{10^2 N/m}$$

$$V_{máx} = k\frac{x_{máx}^2}{2}$$

$$\boxed{V_{máx} = 10^2\frac{N}{m} \cdot \frac{(0'1\,m)^2}{2} = 0'5\,J}$$

b) ¿ Velocidade do conxunto xusto despois do choque ?

Conservación da enerxía :

$$E_{x\,máx\,(2)} = E_{\substack{\text{xusto despois}\\ \text{do choque}}\,(1)}$$

$$T_1 + \cancelto{0}{V_1}{}_{(x=0)} = \cancelto{0}{T_2}{}_{(v_2=0)} + V_2$$

$$T_{máx} = V_{máx} \longrightarrow \frac{1}{2}(M+m)\,v^2 = 0'5\,J$$

$$v = \sqrt{\frac{0'5\,J \cdot 2}{1\,kg}} = 1\,m/s$$

c) ¿ $V_{proxectil}$ xusto antes da colisión ?

Conservación do momento lineal: $\vec{P}_{antes} = \vec{P}_{despois}$

(1D) $\rightarrow$ $P_{antes} = P_{despois}$

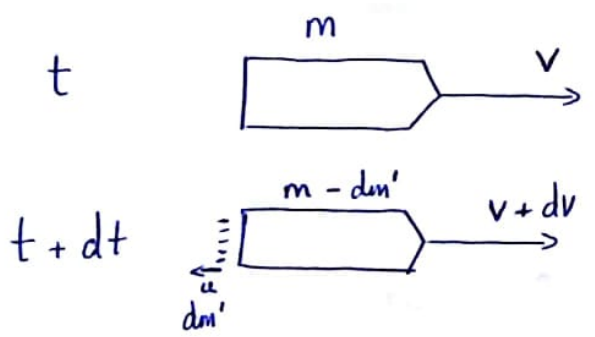
$$P_1 + P_2 = P_{1+2}$$

$$m\, v_p + M\, \cancel{v_B}^{\,0} = (m+M)\, v$$

$$v_p = \frac{(m+M)\, v}{m}$$

$$\boxed{V_p} = \frac{1\,Kg \cdot 1\,m/s}{0'01\,Kg} = \boxed{100\,m/s}$$

(1.15) Usar a conservación do momento para discutir o movemento dun foguete cuxa masa diminúe ao ir expulsando os produtos da combustión do fuel:



$$\boxed{F = \frac{dp}{dt}}$$

$\hookrightarrow$ Forza externa sobre o foguete

$$dp = P_{en\ t+dt} - P_{en\ t}$$

$u \equiv$ velocidade do gas expulsado con respecto do foguete

$$P_{en\ t+dt} = (m - dm')(v + dv) + (v + dv - u)\, dm'$$

$$P_{en\ t} = m\, v$$

$$P_{Total} = P_{foguete} + P_{Gas}$$

(1D $\rightarrow$ podemos prescindir dos vectores)

$$dp = (m - dm')(v + dv) + (v + dv - u)\, dm' - m\, v =$$

$$= mv + m\,dv - \cancel{v\,dm'} - \cancel{dv\,dm'} + \cancel{v\,dm'} + \cancel{dv\,dm'} -$$

$$- u\, dm' - \cancel{mv} = m\, dv - u\, dm'$$

$$\frac{dp}{dt} = m\,\frac{dv}{dt} - u\,\frac{dm'}{dt} \quad , \quad \frac{dm'}{dt} = -\frac{dm}{dt}$$

$$F = \frac{dp}{dt} = m\,\frac{dv}{dt} + u\,\frac{dm}{dt} \longrightarrow$$

Ecuación do movemento do foguete:

$$\boxed{m\frac{dv}{dt} = F_{ext} - u\frac{dm}{dt}}$$

Ⓘ <u>No espacio vacío</u>: $F_{ext} = 0$

$$m\frac{dv}{dt} = -u\frac{dm}{dt}$$

$$m\,dv = -u\,dm$$

$$\int_{v_0}^{v} dv = -u\int_{m_0}^{m}\frac{dm}{m},$$

(Supoñendo $u = cte$)

$$v - v_0 = -u\left(\ln m\right]_{m_0}^{m}$$

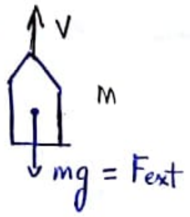
$$v - v_0 = u(\ln m_0 - \ln m)$$

$$\boxed{v(m) = v_0 + u\ln\left(\frac{m_0}{m}\right)}$$

⊛ Supoñendo $m = m_0 - \alpha t$

$$\boxed{v(t) = v_0 + u\ln\left(\frac{m_0}{m_0 - \alpha t}\right)}.$$

Ⓘⓘ <u>En Gravidade</u>:



$$m\frac{dv}{dt} = -mg - u\frac{dm}{dt}$$

⊛ Supoñendo u cte e $(m = m_0 - \alpha t)$ $\frac{dm}{dt} = -\alpha$

$$(m_0 - \alpha t)\frac{dv}{dt} = -(m_0 - \alpha t)g + u\alpha$$

$$\int_{v_0}^{v} dv = \int_0^t \left[-g + \frac{u\alpha}{m_0 - \alpha t}\right]dt$$

$$v - v_0 = \left(-gt - u\ln(m_0 - \alpha t)\right]_0^t =$$

$$= -gt + u\ln\left(\frac{m_0}{m_0 - \alpha t}\right)$$

$$\boxed{v(t) = v_0 - gt + u\ln\left(\frac{m_0}{m_0 - \alpha t}\right)}$$

(16) m $F(x) = -F_0 \sinh(ax)$

$a > 0$

a) ¿ A forza é conservativa ?

Si, porque é unha forza nunha dimensión que depende únicamente da posición

b) Representación de $V(x)$:

F conservativa $\Rightarrow \exists V / \; F(x) = -\frac{dV}{dx}$

$$\int dV = -\int F(x)\,dx$$

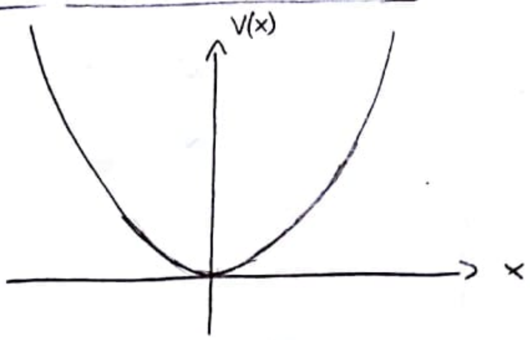
$$\boxed{V} = \int F_0 \sinh(ax)\,dx = \boxed{\frac{F_0}{a}\cosh(ax) + C}$$

Se tomamos $V(0) = 0 \rightarrow V(0) = \frac{F_0}{a}\underbrace{\cosh(0)}_{1} + C = \frac{F_0}{a} + C = 0$

$$\boxed{C = -\frac{F_0}{a}}$$

$$\boxed{V(x) = \frac{F_0}{a}\left[\cosh(ax) - 1\right]}$$

$\cosh x = \frac{e^{x} + e^{-x}}{2}$



c) Existe un punto de equilibrio estable ?

Si, en $x = 0$, por ser un mínimo da enerxía pontencial

$$\left.\frac{dV}{dx}\right|_{x=0} = 0$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=0} > 0$$

Calcular a frecuencia das oscilacións pequenas : $\omega = \sqrt{\frac{k}{m}}$

Taylor de $V(x)$ en torno a $(x=0)$:

$\frac{dV}{dx} = F_0 \sinh(ax)$ $\qquad \frac{d^2V}{dx^2} = F_0\, a \cosh(ax)$

$$V(x) \simeq V(x=0) + \left.\frac{dV}{dx}\right|_{x=0} x + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x=0} x^2$$

$$\left.\frac{dV}{dx}\right|_{x=0} = 0 \qquad \frac{d^2V}{dx^2} = F_0 a$$

$$V(x) \simeq 0 + 0x + \frac{1}{2} F_0 a\, x^2 = \frac{1}{2} F_0 a\, x^2$$

$$F(x) \simeq -kx \ , \qquad F(x) = -\frac{dV}{dx}$$

$$V(x) = -\int F(x)\,dx = k\frac{x^2}{2} + \cancel{cte} \rightarrow 0 \quad (V(0)=0)$$

$$\boxed{k = F_0 a} \qquad \omega = \sqrt{\frac{k}{m}} \ , \quad \boxed{\omega = \sqrt{\frac{F_0 a}{m}}}$$

d) A que distancia deixa de ser adecuada a aproximación do mov. harmónico?

Erro relativo aceptado $\longrightarrow$ 10 %

$$V(x) \simeq \frac{1}{2} F_0 a x^2 \quad (\text{orden } 2)$$

$$V(x) \simeq \frac{1}{2} F_0 a x^2 + \frac{1}{24} F_0 a^3 x^4 \quad (\text{orden catro})$$

← erro cometido

$$\frac{1}{24} \cancel{F_0}\, a^{\overset{2}{\cancel{3}}} x^4 = 0'1 \ \frac{1}{2} \cancel{F_0}\, \cancel{a} x^2$$

$$\frac{a^2 x^2}{12} = 0'1 \qquad x = \frac{\sqrt{1'2}}{a} \rightarrow \text{Deixa de ser unha boa aproximación}$$

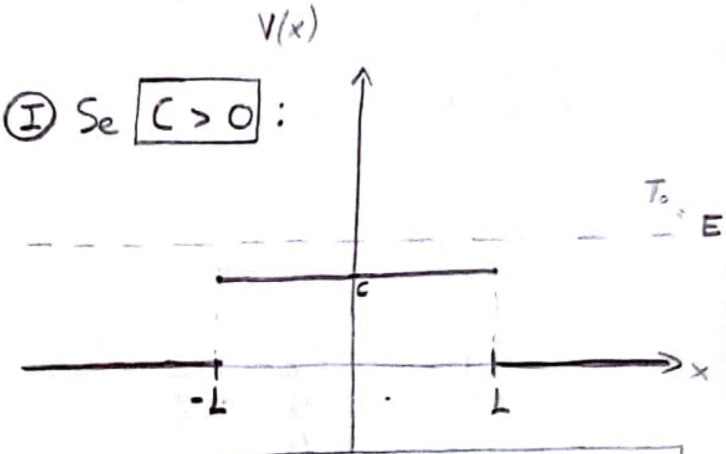
(1.17) m

$$\begin{cases} V(x) = \underset{(cte)}{C} & \text{se } |x| \leq L \\ V(x) = 0 & \text{no resto do espazo} \end{cases}$$

Inicialmente $x_0 << -L$ con $v_0 \rightarrow \boxed{T_0 = \frac{1}{2} v_0^2 m}$

move-se cara a dereita

Ⅰ Se $\boxed{C > 0}$:



a) Representar $V(x)$

Rango de enerxías que permite o movemento descrito?

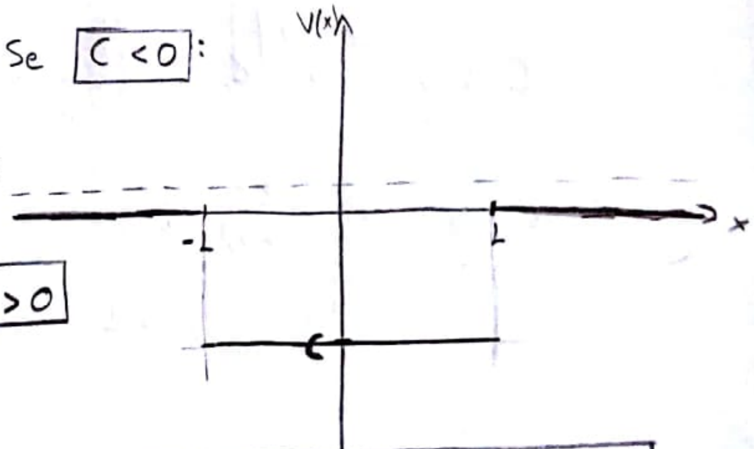
Tendo en conta que $T_0 > 0$

$\rightarrow \boxed{E = T_0 > C}$

$\boxed{E > C}$

Diminúe a velocidade entre -L e L

Ⅱ Se $\boxed{C < 0}$:



(porque $E = T_0 > 0$) $\boxed{E > 0}$

Aumenta a velocidade entre -L e L

b) ¿ Velocidade no extremo da dereita?

A' dereita de L $v = v_0$ por conservación da enerxía:

$$T_0 + \cancel{V_0}^{0} = T(x>L) + \cancel{V(x>L)}^{0}$$

$$T_0 = T(x>L) \rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} m v^2$$

$$\boxed{v = v_0}$$

c) Comparar o tempo que tarda en recorrer o espazo de esquerda a dereita co caso de $C=0$.

Só inflúe o tramo entre L e $-L$. $(2L)$

$$E = T_0 = T + V = \frac{1}{2} m v^2 + C = \frac{1}{2} m v_0^2$$

$$\frac{1}{2} m v^2 = \frac{1}{2} m v_0^2 - C$$

$$v = \sqrt{v_0^2 - \frac{2C}{m}}$$

$$t(C) = \frac{2L}{\sqrt{v_0^2 - \frac{2C}{m}}} = \frac{2L}{v_0 \sqrt{1 - \frac{2C}{m v_0^2}}}$$

$$t(0) = \frac{2L}{v_0}$$

$$T_0 = E > C$$

$$\frac{m_0 v_0^2}{2} = E > C$$

$$\frac{m_0 v_0^2}{2C} > 1$$

$$\boxed{1 > \frac{2C}{m_0 v_0^2}}$$

A raíz existe sempre

$$Z = t(C) - t(0) = \frac{2L}{v_0}\left[\frac{1}{\sqrt{1 - \frac{2C}{m v_0^2}}} - 1\right]$$

$$\boxed{C < 0 \Rightarrow Z < 0} \longrightarrow \text{adianto}$$

$$C < 0 \Rightarrow \frac{2C}{m v_0^2} < 0$$

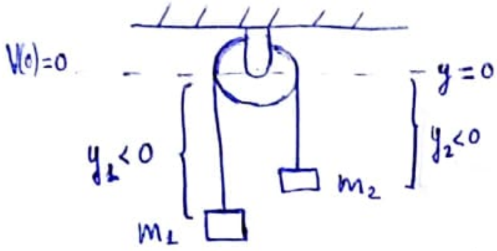
$$1 - \frac{2C}{m v_0^2} > 1 \Longrightarrow Z < 0$$

$$\boxed{C > 0 \Rightarrow Z > 0} \longrightarrow \text{atraso}$$

$$C > 0 \Rightarrow \frac{2C}{m v_0^2} > 0$$

$$1 - \frac{2C}{m v_0^2} < 1 \Longrightarrow Z > 0$$

1.18 Escribir a enerxía total das masas da máquina de Atwood e obter a ecuación do movemento derivando $E = cte$.



masa da polea } desprezables
masa da corda }

corda inextensible (L)

Ligadura: $y_1 + y_2 = L = cte$

$\frac{d}{dt}\left(\dot{y}_1 + \dot{y}_2 = 0 \rightarrow \dot{y} = \dot{y}_1 = -\dot{y}_2\right.$

$\frac{d}{dt}\left(\ddot{y}_1 + \ddot{y}_2 = 0 \rightarrow \ddot{y} = \ddot{y}_1 = -\ddot{y}_2\right.$

$$E = T_1 + V_1 + T_2 + V_2 = \frac{1}{2} m_1 \dot{y}_1^2 + m_1 g y_1 + \frac{1}{2} m_2 \dot{y}_2^2 + m_2 g y_2 =$$

$$= \frac{1}{2} \dot{y}^2 (m_1 + m_2) + m_1 g y_1 + m_2 g y_2 = cte$$

$\frac{d}{dt}$

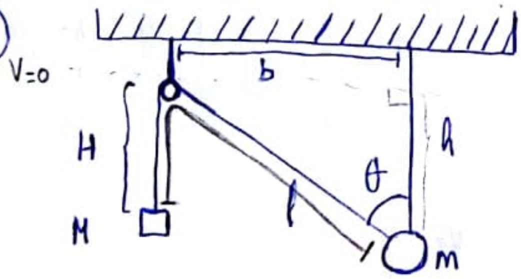
$$\dot{y}\ddot{y}(m_1 + m_2) + m_1 g \dot{y}_1 + m_2 g \dot{y}_2 = 0$$

$$\dot{y}\left[\ddot{y}(m_1 + m_2) + m_1 g - m_2 g\right] = 0$$

$$(m_1 + m_2)\ddot{y} + (m_1 - m_2) g = 0$$

$$\boxed{\ddot{y} = \frac{m_2 - m_1}{m_1 + m_2} g}$$

1.19



$l = cte$

masa da polea desprezable

a) Enerxía potencial en función de θ, b e l

$V_M = -MgH \qquad V_m = -mgh$

$V = V_M + V_m = -g(MH + mh)$

$\sin\theta = \frac{b}{l - H}$

$\tan\theta = \frac{b}{h}, \quad h = b/\tan\theta$

$l - H = b/\sin\theta$

$\boxed{H = l - \frac{b}{\sin\theta}}$

$V = -g\left(Ml - \frac{Mb}{\sin\theta} + \frac{mb}{\tan\theta}\right), \quad \boxed{V(\theta) = \frac{gb}{\sin\theta}(M - m\cos\theta) - Mgl}$

$\theta \in (0, \pi/2)$

b) Atopar as posicións de equilibrio e discutir para que valores das masas existen e a súa estabilidade.

$$V(\theta) = \frac{gb}{\sin\theta}(M - m\cos\theta) - Mg\ell$$

$$\frac{dV}{d\theta} = gb\left[-\frac{\cos\theta}{\sin^2\theta}(M - m\cos\theta) + \frac{1}{\sin\theta}(m\sin\theta)\right] =$$

$$= gb\left[m - \frac{\cos\theta}{\sin^2\theta}(M - m\cos\theta)\right]$$

$$\frac{dV}{d\theta} = 0 \,, \Rightarrow gb\left[\frac{m\sin^2\theta_0 - M\cos\theta_0 + m\cos^2\theta_0}{\sin^2\theta_0}\right] = 0$$

$$m\sin^2\theta_0 - M\cos\theta_0 + m\cos^2\theta_0 = 0$$

$$m\underbrace{(\sin^2\theta_0 + \cos^2\theta_0)}_{1} = M\cos\theta_0$$

$$\boxed{\cos\theta_0 = \frac{m}{M}} \longrightarrow \text{Puntos de equilibrio}$$

(coinciden cos extremos da enerxía potencial)

$\boxed{M > m}$

$$\frac{d^2V}{d\theta^2} = gb\left[\frac{-m2\cos\theta\sin\theta\cdot\sin^2\theta - 2\sin\theta\cos\theta\cos^2\theta\, m}{\sin^4\theta} + \frac{+M\sin\theta\sin^2\theta + M\cos\theta\, 2\sin\theta\cos\theta}{\sin^4\theta}\right]$$

$$= gb\left[\frac{-m\sin 2\theta\,[\sin^2\theta + \cos^2\theta]}{\sin^4\theta} + \frac{M\sin\theta\,[\sin^2\theta + \cos^2\theta\, 2]}{\sin^4\theta}\right] =$$

$$= gb\left[\frac{M\sin\theta\,[1 + \cos^2\theta] - m\sin 2\theta}{\sin^4\theta}\right] = gb\left[\frac{M(1+\cos^2\theta) - 2m\cos\theta}{\sin^3\theta}\right]$$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=\theta_0} = gb\left[\frac{M(1+\cos^2\theta_0) - 2m\cos\theta_0}{\sin^3\theta_0}\right]$$

O signo será o de $M(1+\cos^2\theta_0) - 2m\cos\theta_0 =$

$$= M\left(1 + \frac{m^2}{M^2}\right) - 2\frac{m^2}{M} =$$

$$= M + \frac{m^2}{M} - 2\frac{M^2}{M} = M - \frac{m^2}{M}$$

$$= \frac{1}{M}(M^2 - m^2) > 0 \rightarrow \text{mínimo} \quad \text{punto de eq. estable}$$

$$\theta \in \left(0, \frac{\pi}{2}\right] \Rightarrow \boxed{\begin{matrix}\sin\theta > 0 \\ \cos\theta > 0\end{matrix}}$$

$$\cos\theta_0 = \frac{m}{M} \Rightarrow M > m$$

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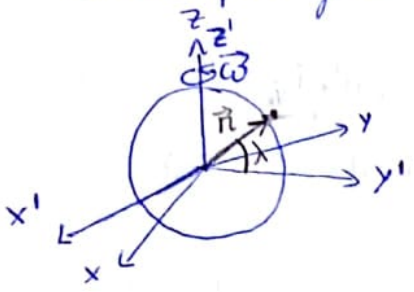
(20) Desviación dunha chumbada respecto á vertical en función da latitude polo efecto centrífugo da rotación da Terra.



$$|\vec{r}| \approx |\vec{R}_T|$$
$$\vec{r} = R_T \underbrace{\left(\cos\lambda\, \hat{y}' + \sin\lambda\, \hat{z}\right)}_{\hat{r}}$$
$$\vec{\omega} = \omega\, \hat{z}$$

Ley de Newton. S R. non inerciais:

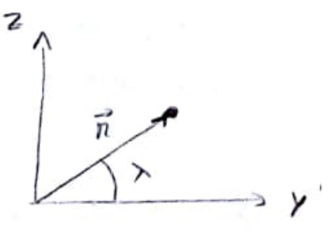
$$m\ddot{\vec{r}} = \vec{F}_e - m\ddot{\vec{R}} - m\dot{\vec{\omega}} \times \vec{r} - 2m(\vec{\omega} \times \dot{\vec{r}}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

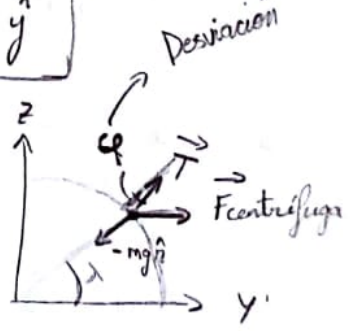
$$\boxed{m\ddot{\vec{r}} = \vec{T} - mg\hat{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})}$$

↳ Forza centrífuga

$$-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -m\left[\vec{\omega} \cdot (\vec{\omega} \cdot \vec{r}) - \vec{r} \cdot (\vec{\omega} \cdot \vec{\omega})\right] =$$
$$= -m\left[\vec{\omega}\,(\omega \sin\lambda R_T) - \vec{r}\,(\omega^2)\right] =$$
$$= -m\left[\omega^2 R_T \sin\lambda\, \hat{z} - \omega^2 R_T \cos\lambda\, \hat{y} - \omega^2 R_T \sin\lambda\, \hat{z}\right] =$$
$$= m\omega^2 R_T \cos\lambda\, \hat{y}$$

$$\boxed{\vec{F}_{centrífuga} = m\omega^2 R_T \cos\lambda\, \hat{y}}$$





Repouso $\rightarrow \sum \vec{F} = 0$

$$\vec{P} + \vec{T} + \vec{F}_{cent} = 0$$

$$\vec{P} = -mg\hat{r} = -mg\cos\lambda\, \hat{y} - mg\sin\lambda\, \hat{z}$$
$$\vec{T} = T_y\, \hat{y} + T_z\, \hat{z}$$
$$\vec{F}_{centrífuga} = m\omega^2 R_T \cos\lambda\, \hat{y}$$

$$\hat{y}\left[-mg\cos\lambda + T_y + m\omega^2 R_T \cos\lambda\right] + \hat{z}\left[-mg\sin\lambda + T_z\right] = 0 \Rightarrow \begin{cases} T_y = mg\cos\lambda - m\omega^2 R_T \cos\lambda \\ T_z = mg\sin\lambda \end{cases}$$

$$\vec{T} = T_y\,\hat{y} + T_z\,\hat{z}$$

$$\begin{cases} T_y = T\cos(\varphi+\lambda) = mg\cos\lambda - m\omega^2 R_T\cos\lambda \\ T_z = T\sin(\varphi+\lambda) = mg\sin\lambda \end{cases}$$

↳ dividiendo

$$\tan(\varphi+\lambda) = \frac{g\sin\lambda}{g\cos\lambda - \omega^2 R_T\cos\lambda}$$

$$\tan(\varphi+\lambda) = \frac{\tan\varphi + \tan\lambda}{1-\tan\varphi\cdot\tan\lambda}$$

$$\tan(\varphi+\lambda) = \frac{g}{g-\omega^2R_T}\tan\lambda$$

$$\tan(\varphi+\lambda) = \frac{1}{1-\frac{\omega^2R_T}{g}}\tan\lambda \overset{⊛}{=} \frac{\tan\lambda}{1-\beta}$$

$$⊛\ \boxed{\beta = \frac{\omega^2R_T}{g}}$$

$$\frac{\tan\varphi + \tan\lambda}{1-\tan\varphi\tan\lambda} = \frac{\tan\lambda}{1-\beta}$$

$$(\tan\varphi+\tan\lambda)(1-\beta) = \tan\lambda - \tan\varphi\tan^2\lambda$$

$$(1-\beta+\tan^2\lambda)\tan\varphi = (1-(1-\beta))\tan\lambda$$

$$\boxed{\tan\varphi = \frac{\beta\tan\lambda}{1-\beta+\tan^2\lambda}}$$

$$\beta \approx \frac{(0'7\,10^{-4})^2\,6370\cdot10^3}{10} \approx 0'0032$$

Despreciable frente a 1

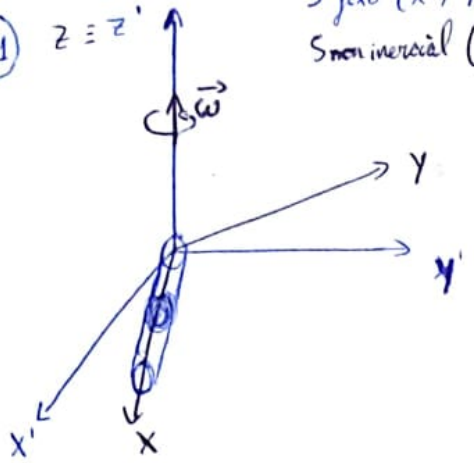
φ pequeño $\longrightarrow$ $\tan\varphi \approx \varphi$ Aproximación

$1-\beta \approx 1$ Aproximación

$$\varphi \approx \frac{\beta\tan\lambda}{1+\tan^2\lambda} = \frac{\beta\tan\lambda}{\frac{1}{\cos^2\lambda}} = \beta\sin\lambda\cos\lambda = \frac{\beta}{2}\sin 2\lambda$$

$$\boxed{\varphi \approx \frac{\omega^2R_T}{2g}\sin 2\lambda}$$

(21) $z \equiv z'$ S fixo (x', y', z')
S non inercial (X, Y, Z)



Tubo lonxitude L
xira con $\vec{\omega}$ uniforme
↳ eixo vertical perpendicular nun extremo

Partícula m no interior inicialmente en repouso respecto do tubo a L_0 do eixo
↓
v de saída ?
t de saída ?
forza do tubo sobre m ?

Lei de Newton sistemas non inerciais:

$$m \cdot \ddot{\vec{r}} = \vec{F}_{ext} - m\cancel{\ddot{\vec{R}}}^{\,0} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v} - m\cancel{\dot{\vec{\omega}}}^{\,0}\,\vec{r}$$

No sistema rotante:
$$\begin{cases} \vec{r} = x\,\hat{x} \\ \vec{v} = \dot{\vec{r}} = \dot{x}\,\hat{x} \\ \vec{a} = \ddot{\vec{r}} = \ddot{x}\,\hat{x} \\ \vec{\omega} = \omega\,\hat{z} \end{cases}$$

Forza centrífuga:
$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & 0 & 0 \end{vmatrix} = \omega x\,\hat{y}$$

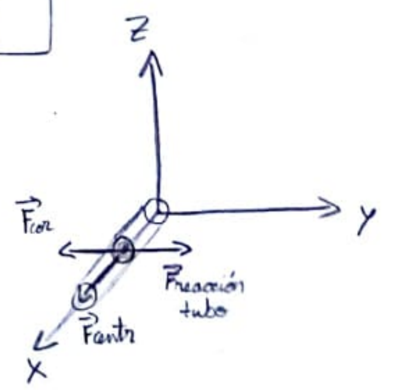
$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ 0 & \omega x & 0 \end{vmatrix} = -\omega^2 x\,\hat{x}$$

$$\boxed{\vec{F}_{centr} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = m\omega^2 x\,\hat{x}}$$

Forza de coriolis:
$$\vec{\omega}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ \dot{x} & 0 & 0 \end{vmatrix} = \omega\dot{x}\,\hat{y}$$

$$\boxed{\vec{F}_{coriolis} = -2m\vec{\omega}\times\vec{v} = -2m\omega\dot{x}\,\hat{y}}$$

$$\boxed{m\ddot{\vec{r}} = \vec{F}_{ext} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}}$$



↳ Compoñentes: → $\hat{x}$: $m\ddot{x} = m\omega^2 x$

$$\boxed{\ddot{x} = \omega^2 x}$$

$\rightarrow \hat{y}:$ $\qquad 0 = F_{reacción} - 2m\omega\dot{x}$

$$\boxed{F_{reacción} = 2m\omega\dot{x}} \longrightarrow \boxed{\vec{F} = 2m\omega\dot{x}\,\hat{y}}$$

Forza que exerce o tubo sobre m

$$\ddot{x} = \omega^2 x$$

$\ddot{x} - \omega^2 x = 0 \longrightarrow$ EDLH coeficientes ctes

$$x = Ce^{\lambda t}$$
$$\ddot{x} = C\lambda^2 e^{\lambda t}$$

$$C\lambda^2 e^{\lambda t} - \omega^2 C e^{\lambda t} = 0$$

$$\lambda^2 - \omega^2 = 0, \qquad \lambda = \pm\sqrt{\omega^2} = \pm\omega$$

Solucións particulares : $\begin{cases} x_1(t) = e^{\omega t} \\ x_2(t) = e^{-\omega t} \end{cases}$

$\downarrow$

Solución xeral :
$$x(t) = A e^{\omega t} + B e^{-\omega t}$$
$$\dot{x}(t) = A\omega e^{\omega t} - B\omega e^{-\omega t}$$

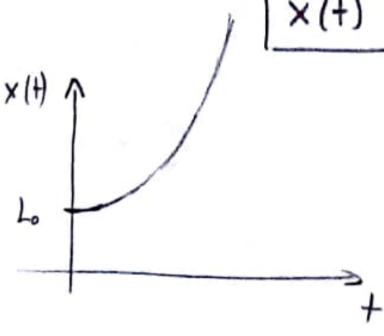
Condicións iniciais :

$$x(0) = A e^{0\omega} + B e^{-\omega 0} = \boxed{A + B = L_0}$$
$$\dot{x}(0) = A\omega - B\omega = \boxed{\omega(A - B) = 0}$$

$$\begin{cases} A + B = L_0 \\ \omega(A-B) = 0 \end{cases} \longrightarrow \omega \neq 0 \Rightarrow A - B = 0$$
$$\boxed{A = B} \longrightarrow \boxed{A = B = \frac{L_0}{2}}$$

$$x(t) = \frac{L_0}{2}\left(e^{\omega t} + e^{-\omega t}\right) = L_0\left(\frac{e^{\omega t} + e^{-\omega t}}{2}\right) = L_0 \cosh(\omega t)$$

$$\boxed{x(t) = L_0 \cosh(\omega t)} \qquad \boxed{v(t) = \omega L_0 \sinh(\omega t)}$$



Sexa t_f o instante no que a bola sae pelo extremo do tubo :

(1) $x(t_f) = L = L_0 \cosh(\omega T_f) = L_0 \dfrac{e^{\omega T_f} + e^{-\omega T_f}}{2}$

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$$L = L_0 \frac{e^{\omega t_f} + e^{-\omega t_f}}{2}$$

$$(2L - L_0 e^{\omega t_f} - L_0 e^{-\omega t_f} = 0) \times e^{\omega t_f}$$

$$2L e^{\omega t_f} - L_0 (e^{\omega t_f})^2 - L_0 = 0$$

$$e^{\omega t_f} = \frac{-2L \pm \sqrt{4L^2 - 4L_0^2}}{-2L_0} = \frac{L \mp \sqrt{L^2 - L_0^2}}{L_0}$$

$$\omega t_f = \ln \frac{L \pm \sqrt{L^2 - L_0^2}}{L_0}$$

$$\boxed{t_f = \frac{1}{\omega} \ln \frac{L \pm \sqrt{L^2 - L_0^2}}{L_0}} \longrightarrow \text{tempo de saída}$$

$$\boxed{V(t) = \omega L_0 \sinh(\omega t)}$$

$$\boxed{V(t_f)} = \omega L_0 \frac{e^{\ln \frac{L + \sqrt{L^2 - L_0^2}}{L_0}} - e^{-\ln \frac{L + \sqrt{L^2 - L_0^2}}{L_0}}}{2} =$$

$$= \frac{\omega L_0}{2} \left[\frac{L + \sqrt{L^2 - L_0^2}}{L_0} - \frac{L_0}{L + \sqrt{L^2 - L_0^2}} \right] = \frac{\omega \cancel{L_0}}{2} \left[\frac{(L + \sqrt{L^2 - L_0^2})^2 - L_0^2}{\cancel{L_0}(L + \sqrt{L^2 - L_0^2})} \right] =$$

$$= \frac{\omega}{2} \left[\frac{\cancel{2}L^2 - \cancel{2}L_0^2 + \cancel{2}L\sqrt{L^2 - L_0^2}}{L + \sqrt{L^2 - L_0^2}} \right] = \omega \frac{\sqrt{L^2 - L_0^2}\,[L + \sqrt{L^2 - L_0^2}]}{L + \sqrt{L^2 - L_0^2}} = \boxed{\omega \sqrt{L^2 - L_0^2}}$$

Outra forma (2)

$$\cosh^2() - \sinh^2() = 1$$

$$\frac{x^2}{L_0^2} - \frac{\dot{x}^2}{\omega^2 L_0^2} = 1 \xrightarrow{x = L} \frac{L^2}{L_0^2} - \frac{\dot{x}^2}{\omega^2 L_0^2} = 1$$

$$\frac{\dot{x}^2}{\omega^2 L_0^2} = \frac{L^2}{L_0^2} - 1 \quad , \quad \dot{x}^2 = \omega^2 L^2 - \omega^2 L_0^2$$

$$\boxed{V = \dot{x} = \omega \sqrt{L^2 - L_0^2}}$$

Outra forma (3)

$$\frac{dv}{dt} = \omega^2 x$$

$$\frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx} v = \omega^2 x$$

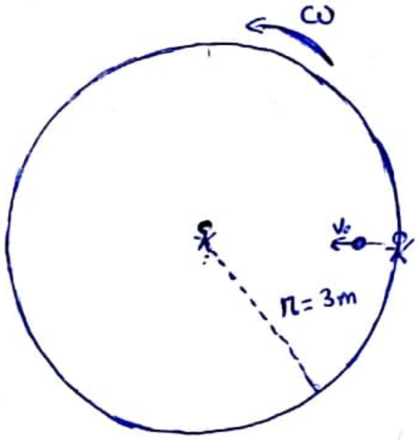
$$\left(\frac{v^2}{2}\right]_{v_0}^{v} = \omega^2\left(\frac{x^2}{2}\right]_{x_0}^{x}$$

$$\frac{1}{2}(v^2 - v_0^2) = \omega^2 \frac{1}{2}(x^2 - x_0^2) \qquad (v_0 = 0)$$

$$v^2 = \omega^2(x^2 - L_0^2) \xrightarrow{x = L} v^2 = \omega^2(L^2 - L_0^2)$$

$$\boxed{v = \omega\sqrt{L^2 - L_0^2}}$$

(22) 2 rapaces ⟶ plataforma circular $r = 3\,m$



$\omega = 2\pi \cdot 12 \cdot \frac{1}{60}\, s^{-1} = \frac{2}{5}\pi$ rad/s

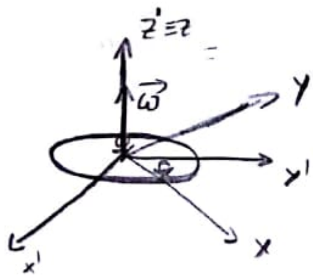
$v_0 = 6\ m/s$ (extremo)

Desprezar a gravidade

Canto deberá estirarse o receptor (no centro)

(desprezamos $\vec{g}$)

$$m\ddot{\vec{r}} = \vec{F}_{ext} - m\ddot{\vec{R}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v} - m\dot{\vec{\omega}}\times\vec{r}$$



$S_{fixo}(x', y', z')$

$S_{rotante}(x, y, z)$

$\vec{r} = x\hat{x} + y\hat{y}$

$\dot{\vec{r}} = \dot{x}\hat{x} + \dot{y}\hat{y}$

$\ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y}$

$\vec{\omega} = \omega\hat{z}$

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = -\omega y\hat{x} + \omega x\hat{y}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} = -\omega^2 x\hat{x} - \omega^2 y\hat{y}$$

$$\boxed{\vec{F}_{centr} = m\omega^2 x\hat{x} + m\omega^2 y\hat{y}}$$

$$\vec{\omega}\times\dot{\vec{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = -\omega\dot{y}\hat{x} + \omega\dot{x}\hat{y}$$

$$\boxed{\vec{F}_{coriolis} = 2m\omega\dot{y}\hat{x} - 2m\omega\dot{x}\hat{y}}$$

Igualando componentes:

$q = x + y\,i$
$\dot{q} = \dot{x} + i\dot{y}$
$\ddot{q} = \ddot{x} + i\ddot{y}$

$\rightarrow \hat{x}: \quad \ddot{x} = \omega^2 x + 2\omega\dot{y}$

$\rightarrow \hat{y}: \quad (\ddot{y} = \omega^2 y - 2\omega\dot{x}) \cdot i$

$$\ddot{x} + i\ddot{y} = \omega^2 (x + iy) + 2\omega \underbrace{(\dot{y} - \dot{x} i)}_{-i(\dot{x} + i\dot{y})}$$

$$\ddot{q} = \omega^2 q - 2\omega i\,\dot{q}$$

$$\boxed{\ddot{q} + 2\omega i\,\dot{q} - \omega^2 q = 0} \rightarrow \text{EDLH } 2^{a} \text{ orden c. ctes}$$

$q = C e^{\lambda t}$

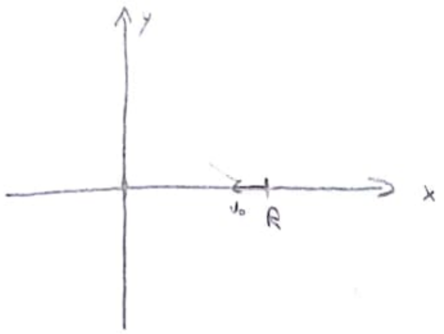
$$C\lambda^2 e^{\lambda t} + 2\omega i\, C\lambda e^{\lambda t} - \omega^2 C e^{\lambda t} = 0$$

$$\lambda^2 + 2\omega i\lambda - \omega^2 = 0$$

$$\lambda = \frac{-\cancel{2}i\omega \pm \sqrt{-4\omega^2 + 4\omega^2}}{\cancel{2}} = -i\omega \pm 0 = -i\omega$$

$$\boxed{q = A \cdot e^{-i\omega t} + B\,t\, e^{-i\omega t}} \quad \ldots$$

Considerando un SR inercial $\longrightarrow$ Non hai forzas externas



$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

$$\dot{\vec{r}} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$$

$$\boxed{m\,\ddot{\vec{r}} = 0} \Longrightarrow \ddot{\vec{r}} = 0 \Rightarrow \dot{\vec{r}} = cte \quad \left| \begin{array}{l} \dot{x} = cte \\ \dot{y} = cte \end{array} \right.$$

$\dot{x} = -v_0$

$\dot{y} = \omega R$

$$\dot{\vec{r}} = -v_0\,\hat{i} + \omega R\,\hat{j}$$

$$\dot{x} = -v_0 \;,\quad \frac{dx}{dt} = -v_0 \;,\quad \int_R^x dx = -v_0\int_0^t dt$$

$$\boxed{x = R - v_0 t}$$

$$\dot{y} = \omega R \;,\quad \frac{dy}{dt} = \omega R \;,\quad \int_0^y dy = \omega R \int_0^t dt$$

$$\boxed{y = \omega R t}$$

$$\vec{r} = (R - v_0 t)\,\vec{i} + \omega R t\,\vec{j}$$

Cando a bola é recibida : $x(t_f) = 0$

$$R - v_0 t_f = 0$$

$$\boxed{t_f = R/v_0}$$

$$y(t_f) = \omega R t_f$$

$$\boxed{y(t_f) = \frac{\omega R^2}{v_0}} \longrightarrow \text{desvío}$$

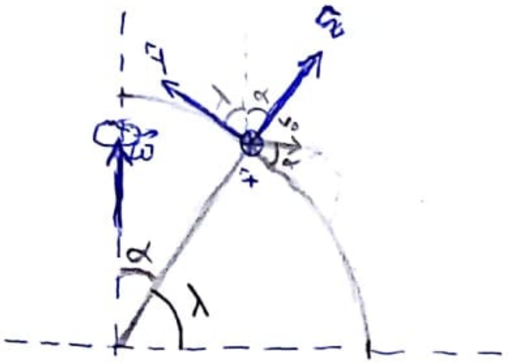
$$y = \frac{\frac{2}{5}\pi\, s^{-1} \cdot (3m)^2}{6\, m s^{-1}} = \frac{3}{5}\pi\, m \simeq \boxed{1'88\ m}$$

1.23 proxectil balístico → lánzase a 500 m/s cara o sur formando 30° ca horizontal

¿ Desviación por coriolis ?

$\lambda = 60° N$ $\alpha = 30°$

↳ latitude



$\vec{\omega} = \omega \sin\alpha \, \hat{y} + \omega \cos\alpha \, \hat{z}$

$\vec{V}_0 = V_0 \sin\alpha \, \hat{z} - V_0 \cos\alpha \, \hat{y}$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\dot{\vec{r}} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$

$\ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$

$$m\ddot{\vec{r}} \simeq -mg\hat{z} - 2m\vec{\omega}\times\dot{\vec{r}}$$

$$\vec{\omega}\times\dot{\vec{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \frac{\omega}{2} & \frac{\omega\sqrt{3}}{2} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = \left(\frac{\omega}{2}\dot{z} - \frac{\omega}{2}\sqrt{3}\dot{y}\right)\hat{x} + \frac{\omega}{2}\sqrt{3}\dot{x}\hat{y} - \frac{\omega}{2}\dot{x}\hat{z}$$

$$\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} = (\omega\dot{z} + \omega\sqrt{3}\dot{y})\hat{x} + (-\omega\sqrt{3}\dot{x})\hat{y} + (\omega\dot{x} - g)\hat{z}$$

↳ Igualando compoñentes :

$$\begin{cases} \ddot{x} = -\omega\dot{z} + \omega\sqrt{3}\dot{y} = \omega(-\dot{z} + \sqrt{3}\dot{y}) \\ \ddot{y} = -\omega\sqrt{3}\dot{x} \\ \ddot{z} = \omega\dot{x} - g \end{cases}$$

Supoñemos traxectoria plana desprezando a desviación en x fronte ao movemento en y e z e desprezamos a curvatura da terra ($z(t_{caída}) = 0$) para calcular o tempo de caída (e $\dot{y}, \dot{z}$).

↳ $\dot{x} \simeq 0$ $\ddot{x} \simeq 0$ $x \simeq 0$

$0 = -\omega\dot{z} + \omega\sqrt{3}\dot{y}$

$\ddot{y} = 0 \longrightarrow \dot{y} \simeq cte = -V_0\cos\alpha$

$\ddot{z} = -g \rightarrow \dot{z} = \dot{z}_0 - gt = v_0\sin\alpha - gt \xrightarrow{(z_0=0)} z(t) = V_0\sin\alpha\, t - \frac{1}{2}gt^2$

$$z(t_{caída}) = v_0 \sin\alpha \, t_{caída} - \frac{1}{2} g \, t_{caída}^2 = 0$$

$$t_c \left(v_0 \sin\alpha - \frac{1}{2} g \, t_c \right) = 0$$

$t_c = 0$ Non vale $(t_c \neq 0)$

$$v_0 \sin\alpha - \frac{1}{2} g \, t_c = 0$$

$$\boxed{\frac{2 v_0 \sin\alpha}{g} = t_c}$$

$$\begin{cases} \ddot{x} = \omega(-\dot{z} + \sqrt{3}\,\dot{y}) \\ \ddot{y} = -\omega\sqrt{3}\,\dot{x} \\ \ddot{z} = \omega \dot{x} - g \end{cases}$$

$$\dot{z} \simeq v_0 \sin\alpha - gt$$
$$\dot{y} \simeq -v_0 \cos\alpha$$

$$\ddot{x} = \omega\left(-v_0 \sin\alpha + gt - \sqrt{3} v_0 \cos\alpha\right) =$$
$$= \omega\left(-v_0/2 - \frac{3}{2} v_0 + gt\right) = \omega(-2v_0 + gt) =$$
$$= -2\omega v_0 + g\omega t$$

$$\dot{x}(t) = -2\omega v_0 t + \frac{1}{2} g \omega t^2 \quad (\dot{x}_0 = 0)$$

$$x(t) = -\omega v_0 t^2 + \frac{1}{6} g \omega t^3 \quad (x_0 = 0)$$

$$x(t_c) = -\omega v_0 \frac{4 v_0^2 \sin^2\alpha}{g^2} + \frac{1}{6} g \omega \frac{8 v_0^3 \sin^3\alpha}{g^3} =$$

$$= -\omega v_0^3 \frac{1}{4} \frac{4}{g^2} + \frac{1}{3} \omega v_0^3 \frac{4 \cdot 1/8}{g^2} =$$

$$= -\frac{\omega v_0^3}{g^2} + \frac{\omega v_0^3}{6 g^2} = \frac{-6 \omega v_0^3 + \omega v_0^3}{6 g^2} = -\frac{5 \omega v_0^3}{6 g^2}$$

$$\boxed{x(t_c) \simeq -78'9 \text{ m}} \longrightarrow \boxed{\text{Desvíase 79 m cara o oeste}}$$

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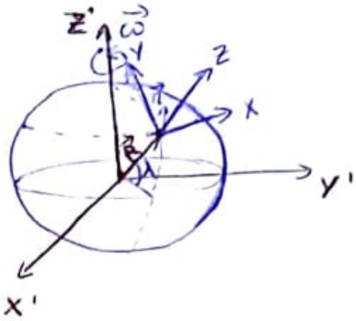
(24) Desviación pola aceleración de Coriolis dun corpo que cae dende $h = 100\,m$ nun lugar da terra con latitude $\lambda_0 = 45^\circ N$.

$\lambda \equiv$ latitude



$$\vec{r} + \vec{R} \simeq \vec{R}$$

$|\vec{r}| = 100\,m$

$|\vec{R}| \simeq 6380\,Km$

$$m\ddot{\vec{r}} = m\vec{g}_0 - m\ddot{\vec{R}} - \cancel{m\dot{\vec{\omega}} \times \vec{r}}^{\,0} - m\vec{\omega} \times [\vec{\omega} \times \vec{r}] - 2m(\vec{\omega} \times \dot{\vec{r}})$$

$$\frac{d\vec{R}}{dt} = \cancel{\frac{d'\vec{R}}{dt}}^{\,0} + \vec{\omega} \times \vec{R} \quad , \quad \frac{d^2\vec{R}}{dt^2} = \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

$$m\ddot{\vec{r}} = m\vec{g}_0 - m\vec{\omega} \times (\vec{\omega} \times \underbrace{(\vec{r} + \vec{R})}_{\simeq \vec{R}}) - 2m(\vec{\omega} \times \dot{\vec{r}})$$

$$m\ddot{\vec{r}} \simeq m\vec{g}_0 - m\vec{\omega} \times (\vec{\omega} \times \vec{R}) - 2m(\vec{\omega} \times \dot{\vec{r}})$$

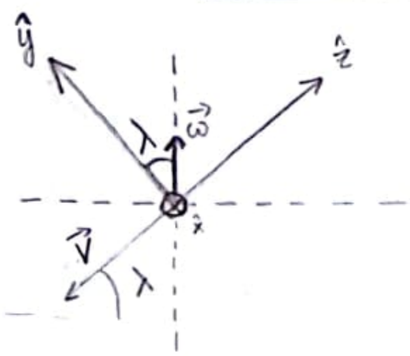
$$|\vec{\omega} \times (\vec{\omega} \times \vec{R})| = \omega^2 R \sin\theta \leq \omega^2 R \simeq 0'033\ m/s^2$$

$$\omega^2 \simeq 5'3\ 10^{-9}\ s^{-2}$$

$$R \simeq 6'371\ 10^6\ m$$

$|\vec{g}| \ggg |\vec{\omega} \times (\vec{\omega} \times \vec{R})| \longrightarrow$ Desprezamos a forza centrífuga

$$\boxed{m\ddot{\vec{r}} \simeq m\vec{g}_0 - 2m(\vec{\omega} \times \dot{\vec{r}})}$$



$$\vec{\omega} = \omega\cos\lambda\ \hat{y} + \omega\sin\lambda\ \hat{z}$$

$$\vec{v} = \dot{x}\ \hat{x} + \dot{y}\ \hat{y} + \dot{z}\ \hat{z} \simeq \dot{z}\ \hat{z}$$

$$\dot{x}\ ,\ \dot{y} \lll \dot{z}$$

$$\vec{F}_{cor} = -2m\vec{\omega} \times \vec{v} = -2m \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & 0 & \dot{z} \end{vmatrix} = -2m\omega\dot{z}\cos\lambda\ \hat{x}$$

Componentes da ecuación : $m\ddot{\vec{r}} = m\vec{g}_0 - 2m(\vec{\omega}\times\vec{v})$

En x : $m\ddot{x} = -2m\omega\dot{z}\cos\lambda$

En y : $m\ddot{y} = 0$

En z : $m\ddot{z} = -mg_0 \longrightarrow \ddot{z} = -g_0$

$\boxed{\dot{z} = -g_0 t}$ $\dot{z}_0 = 0$ (cae)

$\boxed{z = h - \frac{1}{2}g_0 t^2}$ $z_0 = h$

$\ddot{x} = 2\omega g_0 t\cos\lambda$

$\dot{x} = \omega g_0 \cos\lambda\, t^2$ $(\dot{x}_0 = 0)$

$\boxed{x(t) = \frac{\omega g_0 \cos\lambda}{3} t^3}$ $(x_0 = 0)$

Desviación : $x(t_c)$ $t_c \equiv$ tempo da caída

$z(t_c) = 0 \longrightarrow z = h - \frac{1}{2}g_0 t_c^2 = 0$

$\boxed{t_c = \sqrt{\frac{2h}{g_0}}}$

$$\boxed{x(h,\lambda) = \frac{1}{3}\omega g_0 \cos\lambda \left(\frac{2h}{g_0}\right)^{3/2}}$$

$$x = \frac{1}{3}\, 7'3\cdot 10^{-5}\, s^{-1}\cdot 9'8\, m/s^2 \cos 45^\circ \left(\frac{2\cdot 100m}{9'8\, m/s^2}\right)^{3/2} \simeq$$

$\simeq 0'015\, m \longrightarrow \boxed{1'5\ cm}$ (cara o leste)

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Control 2016

m

$F = -F_0 \cdot e^{-x/\lambda}$

$\lambda > 0$

$t_0 = 0$
$x_0 = 0$ $\quad v_0 > 0$

¿ $v(x)$?

2ª Ley de Newton:

$$m \frac{dv}{dt} = -F_0 \cdot e^{-x/\lambda}$$

$$m \frac{dv}{dx} \cdot \frac{dx}{dt} = -F_0 \cdot e^{-x/\lambda}$$

$$m \int_{v_0}^{v} v \cdot dv = -F_0 \int_0^x e^{-x/\lambda} \cdot dx$$

$$\frac{1}{2} m (v^2 - v_0^2) = F_0 \lambda \cdot \left(e^{-x/\lambda} \right]_0^x$$

$$\frac{1}{2} m (v^2 - v_0^2) = F_0 \lambda \left[e^{-x/\lambda} - 1 \right]$$

$$\boxed{v(x) = \oplus \left[v_0^2 + \frac{2 F_0 \lambda}{m} \left(e^{-x/\lambda} - 1 \right) \right]^{1/2}}$$

En $x = 0$, $v_0 > 0$

Verifica la c.i.

$\underset{(m)}{[x]} = \underset{(m)}{[\lambda]}$ Análisis dimensional ✓

$$\frac{F_0 \lambda}{m} = \frac{Kg\, m\, s^{-2}\, m}{Kg} = m^2 s^{-2}$$

¿ v_∞ ?

$t \to \infty$
$x \to \infty$

$(v(x) > 0 \ \forall x)$

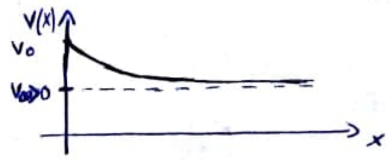
$x \to \infty$

$$v \to v_\infty = \sqrt{v_0^2 - \frac{2 F_0 \lambda}{m}} = \sqrt{v_0^2 - v_e^2}$$

$$v_e^2 = \frac{2 F_0 \lambda}{m}$$

Casos:

Ⓘ $v_0 > v_e \implies v_\infty > 0$



Ⓘⓘ $v_0 = v_e \implies v_\infty = 0$

III $v_0 < v_e \Rightarrow \nexists v_\infty$

No llega a $x \to \infty$, $\exists\, x_m / v(x_m) = 0$

$$v(x_m) = \left[v_0^2 + \frac{2F_0\lambda}{m}\left(e^{-\frac{x_m}{\lambda}} - 1\right)\right]^{1/2} = 0$$

$$v_0^2 = -v_e^2\left(e^{-\frac{x_m}{\lambda}} - 1\right)$$

$$e^{-\frac{x_m}{\lambda}} - 1 = -\frac{v_0^2}{v_e^2}$$

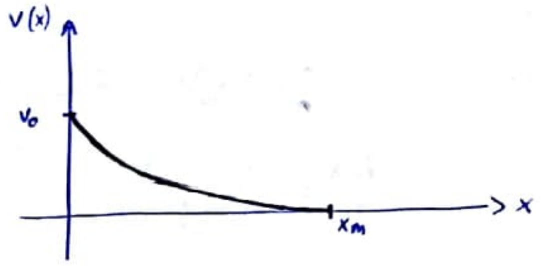
$$e^{-\frac{x_m}{\lambda}} = 1 - \frac{v_0^2}{v_e^2}$$

$$\boxed{x_m = -\lambda \ln\left(1 - \frac{v_0^2}{v_e^2}\right)}$$

Comprobamos:

$$v(x_m) = \left[v_0^2 + v_e^2\left(e^{-\frac{-\lambda\ln\left(1-\frac{v_0^2}{v_e^2}\right)}{\lambda}} - 1\right)\right]^{1/2} =$$

$$= v_0^2 + v_e^2\left(1 - \frac{v_0^2}{v_e^2} - 1\right) = 0 \;\checkmark$$



De otra forma:

$F(x) = -F_0 \cdot e^{-x/\lambda}$ $\rightarrow$ Fuerza dependiente de la posición en una dimensión $\Rightarrow$ siempre conservativa

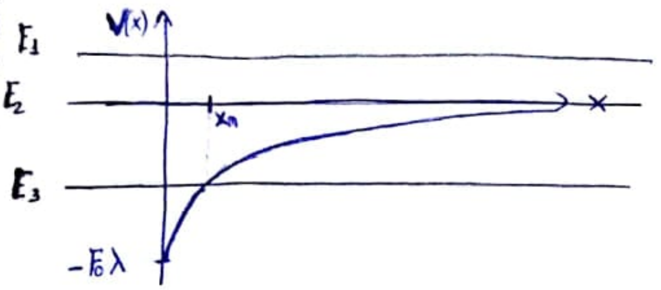
$$F = -\frac{dV}{dx} \quad , \quad dV = -F\,dx$$

$$\int dV = -\int F\,dx$$

$$V(x) = F_0 \int e^{-x/\lambda}\,dx = -\lambda F_0 e^{-x/\lambda} + C$$

$\boxed{x \to \infty \Rightarrow V(x) = 0} \longrightarrow \boxed{C = 0}$ $\qquad \boxed{V(x) = -F_0 \lambda e^{-x/\lambda}}$

$\hookrightarrow 0 = 0 + C$



La masa m solo se puede mover en aquellos puntos x tales que $E \geq V(x)$

E se conserva.

$E = V + T$

Casos:

① $E_1 > V(x) \quad \forall x$

la partícula llega a $x \to \infty$. $E_1^{(\infty)} = \frac{1}{2} m v_\infty^2 + \cancel{V}^{\to 0}$

$$E_1(x=0) = \frac{1}{2} m v_0^2 + V(x=0) = \frac{1}{2} m v_0^2 + (-F_0\lambda)$$

$$E_1(x\to\infty) = E_1(x=0) \quad \text{(Conservación)}$$

$$\frac{1}{2} m v_\infty^2 = \frac{1}{2} m v_0^2 + (-F_0\lambda)$$

$$v_\infty^2 = v_0^2 - \frac{2F_0\lambda}{m}$$

$$v_\infty = \sqrt{v_0^2 - v_e^2}$$

② $E_2 = 0$ $\quad E_2^{(\infty)} = \frac{1}{2} m v_\infty^2$

$$E_2(x\to\infty) = E_2(x=0) = 0$$

$\frac{1}{2} m v_\infty^2 = 0$, $v_\infty = 0 \longrightarrow$ Se para en el infinito

$$\frac{1}{2} m v_\infty^2 = \frac{1}{2} m v_0^2 - F_0\lambda = 0$$

$$v_0^2 = v_e^2 = \frac{2F_0\lambda}{m} \quad (v_0 = v_e)$$

③ $E_3 < 0 \quad (-\lambda F_0 < E_3 < 0)$ $\quad \nexists\, v_\infty \to$ No llega a $x \to \infty$

$x \in [0, x_m]$

$$E_3(x_m) = E_3(x=0)$$

$$E_3(x_m) = \frac{1}{2} m \cancel{v^2(x_m)}^{\to 0} + V(x_m) = -F_0\lambda\, e^{-\frac{x_m}{\lambda}}$$

$$E_3(x_0) = \frac{1}{2} m v_0^2 - F_0 \lambda$$

$$\frac{1}{2} m v_0^2 - F_0 \lambda = - F_0 \lambda e^{-\frac{x_m}{\lambda}}$$

$$\boxed{x_m = - \lambda \ln\left(1 - \frac{V_0^2}{V_e^2}\right)}$$

Tema 2: Formulación Lagrangiana de la Mecánica

2.1 Ligaduras y coordenadas generalizadas:

- **Ligadura**: condición o restricción que limita el movimiento.

a) Holónomas:

Si existe una ecuación que relacione todas las coordenadas de una partícula tal que:

$$f(\underbrace{\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_N}_{N \text{ partículas} \longrightarrow 3N \text{ coordenadas}}, t) = 0$$

$$\left. \begin{array}{l} f_j(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t) = 0 \\ j = 1, 2, \ldots, k \end{array} \right\} \longrightarrow k \text{ ligaduras}$$

$\boxed{3N - k \text{ grados de libertad}}$

1

p.e. $Ax + By + Cz - D = 0$ (la partícula se mueve en el plano)

b) No holónomas:

Cuando las coordenadas no se pueden escribir como holónomas.

✱ $\boxed{\text{no permiten eliminar grados de libertad}}$

- Lineales
- No lineales

p.e. Gas en una caja $0 \leq x \leq L$

2

- a) Escleronomas: no dependen del tiempo (fijas)
- b) Reónomas: dependen del tiempo (móviles)

□ Coordenadas generalizadas:

Cualquier conjunto de parámetros numéricos que determinan de forma unívoca el estado de un sistema con un número finito de grados de libertad en cualquier instante.

q_i , $i = 1, 2, 3, \ldots, n$

→ distancias $(x_i, y_i, z_i, \ldots)$
ángulos $(\theta_i, \varphi_i, \ldots)$
. . .

□ Grados de libertad:

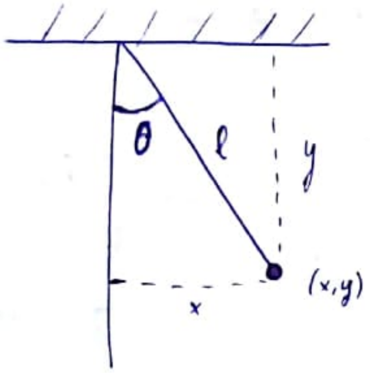
Conjunto de coordenadas generalizadas (q_j) independientes.
Mínimo número de coordenadas generalizadas necesarias para definir el sistema.

→ Ligaduras ⇒ Las variables q_j no son independientes → Reducción del nº de grados de libertad
⇒ Fuerzas asociadas a las ligaduras → Difícil cálculo

$$\boxed{\begin{array}{l} N \text{ partículas en 3D} \longrightarrow 3N \text{ coordenadas} \\ k \text{ ligaduras} \longrightarrow 3N-k \text{ grados de libertad.} \end{array}}$$

$\vec{r}_i(q_j) \longrightarrow$ conjunto propio (sistema de coordenadas generalizadas independientes)

→ Ejemplo 1 Péndulo Simple:



1 partícula → 3D

2 ligaduras $\begin{cases} x^2+y^2=\ell^2 \\ z=0 \end{cases}$

$g.\ell. \Rightarrow 3-2 = \boxed{1\ g.\ell.}$

↓

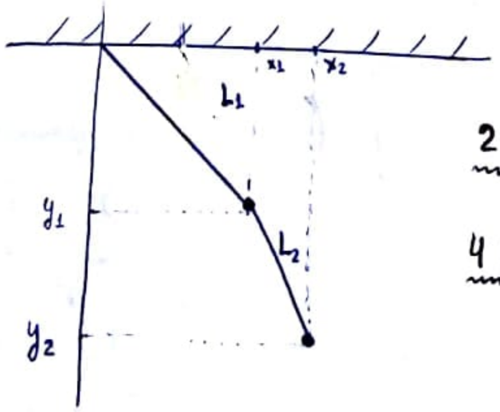
$\boxed{\text{1 coordenada generalizada}}$

$\boxed{q_1=\theta}$ (podríamos coger otra)

⊛ Relación entre coordenadas cartesianas y generalizadas:

$$\begin{cases} x = \ell \sin\theta \\ y = \ell \cos\theta \end{cases} \qquad \theta = \arctan\frac{x}{y}$$

→ Ejemplo 2: Péndulo doble:



2 partículas → 3D

4 ligaduras $\begin{cases} x_1^2+y_1^2 = L_1^2 \\ (x_2-x_1)^2+(y_2-y_1)^2 = L_2^2 \\ z_1 = 0 \\ z_2 = 0 \end{cases}$

⊛ Relación c. cart y generalizadas:

$x_1 = l_1 \sin\theta_1 \quad y_1 = l_1 \cos\theta_1$

$x_2 = x_1 + l_2\sin\theta_2 = l_1\sin\theta_1 + l_2\sin\theta_2$

$y_2 = y_1 + l_2\cos\theta_2 = l_1\cos\theta_1 + l_2\cos\theta_2$

$$\begin{cases} q_1 = \theta_1 = \arctan\frac{x_1}{y_1} \\ q_2 = \theta_2 = \arctan\frac{x_2-x_1}{y_2-y_1} \end{cases}$$

$g.\ell. \Rightarrow 6-4 = \boxed{2\ g.\ell.}$

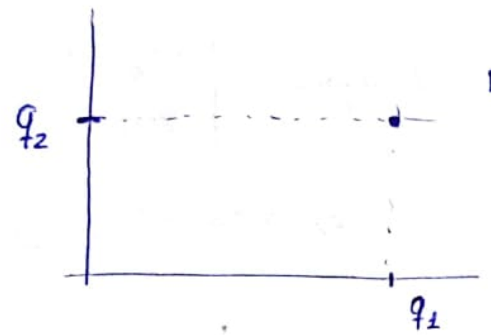
↓

$\boxed{\text{2 coordenadas generalizadas}}$

□ Espacio de configuración:

Espacio constituido por las coordenadas generalizadas cuya dimensión viene dada por $n = 3N - k$.

p.e.
$\boxed{n=2}$



□ Diagrama de fase $\longrightarrow$ Espacio de los $\{q, \dot{q}, \dots\}$

⊛ Si q_j es una coordenada generalizada $\boxed{\dot{q}_j = \frac{dq_j}{dt}}$ es una velocidad generalizada.

$\boxed{2.2}$ Principio de los trabajos virtuales, principio de D'Alembert y ecuaciones de Lagrange:

" Las fuerzas externas que actúan sobre un cuerpo forman un sistema de fuerzas en equilibrio. "

$\boxed{\text{(I) Principio de trabajos virtuales}}$ (sistemas en equilibrio) (estática)

$$\boxed{\begin{array}{c}\vec{F_i} = 0 \\ i = 1, 2, \dots, N\end{array}}$$

Suma de todas las fuerzas que actúan sobre i

□ Desplazamiento virtual:

δq_i:
1. infinitesimal
2. instantáneo ($\delta t = 0$)
3. consistente con las ligaduras

⊛ $\delta \vec{r}_i \rightarrow$ desplazamiento virtual en las coordenadas cartesianas

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La suma de las fuerzas que actúan sobre i es nula:

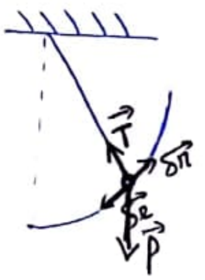
$$\vec{F}_i = 0 \implies \vec{F}_i \cdot \delta\vec{r}_i = 0$$

$i = 1, 2, \ldots, N$

⊛ Descomposición de las fuerzas: $\boxed{\vec{F}_i = \vec{F}_i^{(e)} + \vec{f}_i}$

- $\vec{F}_i$: fuerzas sobre la partícula i
- $\vec{F}_i^{(e)}$: fuerzas aplicadas
- $\vec{f}_i$: fuerzas de ligadura

p.e.



$$\left.\begin{aligned} \vec{F}_i^{(e)} &= \vec{P} \\ \vec{f}_i &= \vec{T} \end{aligned}\right\} \longrightarrow$$ **El trabajo de las fuerzas de ligadura es nulo**

(por ser $\delta\vec{r}_i$ consistentes con las ligaduras)

$$\vec{F}_i \cdot \delta\vec{r}_i = 0, \quad \left(\vec{F}_i^{(e)} + \vec{f}_i\right)\delta\vec{r}_i = 0$$

$$\sum_{i=1}^{N}\left(\vec{F}_i^{(e)} + \vec{f}_i\right)\cdot\delta\vec{r}_i = \sum_{i=1}^{N}\left[\vec{F}_i^{(e)}\cdot\delta\vec{r}_i + \underbrace{\vec{f}_i\cdot\delta\vec{r}_i}_{=0}\right] = 0$$

Principio de trabajos virtuales

$$\boxed{\delta W = \sum_{i=1}^{N}\vec{F}_i^{(e)}\cdot\delta\vec{r}_i = 0}$$

El trabajo virtual llevado a cabo por las fuerzas externas es 0.

⊛ Con $n = 3N - k$ g.l. $\left\{\begin{aligned} \vec{r}_1 &= \vec{r}_1(q_1, \ldots, q_n, t) \\ &\ldots \\ \vec{r}_N &= \vec{r}_N(q_1, \ldots, q_n, t) \end{aligned}\right.$

$$d\vec{r}_i = \sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j + \cancel{\frac{\partial\vec{r}_i}{\partial t}\partial t}^{\,0 \text{ (desplazamiento virtual)}}$$

$$\sum_{i=1}^{N}\vec{F}_i^{(e)}\cdot\sum_{j=1}^{n}\frac{\delta\vec{r}_i}{\delta q_j}\delta q_j = \sum_{j=1}^{n}\delta q_j\overbrace{\left(\sum_{i=1}^{N}\vec{F}_i^{(e)}\frac{\delta\vec{r}_i}{\delta q_j}\right)}^{=0} = 0 \Rightarrow \delta q_i \text{ independientes}$$

$$\boxed{Q_j = \sum_{i=1}^{N} \vec{F}_i^{(e)} \left(\frac{\delta \vec{r}_i}{\delta q_j}\right) \qquad j = 1, \dots, 3N-k}$$ Fuerza Generalizada

II Principio de D'Alembert

(sistemas en movimiento)
(dinámica)

2ª Ley de Newton: $\vec{F}_i = \frac{d\vec{p}_i}{dt} \qquad i = 1, \dots, N$

$$\vec{F}_i - \frac{d\vec{p}_i}{dt} = 0$$

$$\left[\vec{F}_i - \frac{d\vec{p}_i}{dt}\right] \delta\vec{r}_i = 0$$

desplazamiento virtual

$$\vec{F}_i = \vec{F}_i^{(e)} + \vec{f}_i$$

$$\sum_{i=1}^{N} \left[\vec{F}_i^{(e)} + \underbrace{\vec{f}_i}_{} - \frac{d\vec{p}_i}{dt}\right] \underbrace{\delta\vec{r}_i}_{} = 0$$

$\vec{f}_i \cdot \delta\vec{r}_i = 0$

$$\boxed{\delta W = \sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i)\delta\vec{r}_i = 0} \leftarrow \sum_{i=1}^{N} \left[\vec{F}_i^{(e)} - \frac{d\vec{p}_i}{dt}\right] \delta\vec{r}_i = 0$$

$$\delta\vec{r}_i = \sum_{j=1}^{k} \left(\frac{\partial \vec{r}_i}{\partial q_j}\right) \delta q_j$$

$$\sum_{i=1}^{N} \vec{F}_i^{(e)} \cdot \delta\vec{r}_i = \sum_{j=1}^{n} \delta q_j \left[\sum_{i=1}^{N} \vec{F}_i^{(e)} \frac{d\vec{r}_i}{dq_j}\right] =$$

$$= \sum_{j=1}^{n} \delta q_j \, Q_j$$

$$\sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} \delta\vec{r}_i = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \, \delta\vec{r}_i = \sum_{i=1}^{N} \sum_{j=1}^{n} m_i \ddot{\vec{r}}_i \frac{d\vec{r}_i}{dq_j} \delta q_j =$$

$$= \sum_{j=1}^{n} \delta q_j \underbrace{\left[\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right]}_{}$$

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} \left[m_i \frac{d}{dt}\left(\dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right] =$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right] - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \frac{\partial}{\partial q_j}(\dot{\vec{r}}_i) =$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right] - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j} =$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{N} m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right] - \sum_{i=1}^{N} m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_j} =$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2\right)\right) - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2\right) =$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}$$

Tenemos $\longrightarrow$ $\displaystyle\sum_{i=1}^{N} \left[\vec{F}_i^{(e)} - \dot{\vec{P}}_i\right] \delta \vec{r}_i = 0$

$$\left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{N} \vec{F}_i^{(e)} \delta \vec{r}_i = \sum_{j=1}^{n} \delta q_j Q_j \\ \displaystyle\sum_{i=1}^{N} \dot{\vec{P}}_i \delta \vec{r}_i = \sum_{j=1}^{n} \delta q_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] \end{array}\right.$$

$$\boxed{\sum_{j=1}^{n} \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0}$$

$q_j \rightarrow$ coord. generalizadas

[Ppio D'Alembert]

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j} \longrightarrow \text{Ecuación de movimiento del sistema}$$

$j = 1, 2, \ldots, n$

$\rightarrow$ Si: [1] Las fuerzas externas son conservativas $\left(\vec{F}_i^{(e)} = -\vec{\nabla}_i V\right)$

y

[2] El potencial (V) es independiente de las velocidades generalizadas ($\dot{q}_j$) $\left(\frac{\partial V}{\partial \dot{q}_j} = 0\right)$

Entonces:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \rightarrow \text{Ecuaciones de Lagrange o de Euler-Lagrange}$$

$j = 1, 2, \ldots, 3N-k$ (n)

$\boxed{L = T - V}$ Lagrangiana del sistema

Demostración: Partimos de $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$

e $Q_j = \sum_{i=1}^{N} \vec{F}_i^{(e)} \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$

$$Q_j = \sum_{i=1}^{N} \vec{F}_i^{(e)} \left(\frac{\partial \vec{r}_i}{\partial q_j}\right) \overset{\boxed{1}}{=} \sum_{i=1}^{N} (-\vec{\nabla}_i V)\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = -\sum_{i=1}^{N} \frac{\partial V}{\partial \vec{r}_i}\frac{\partial \vec{r}_i}{\partial q_j} =$$

$$= -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{q_j} = -\frac{\partial V}{q_j} \quad , \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}\right) = 0$$

$\boxed{2}$ $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}\right) = 0 \quad , \quad \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \left(\frac{\partial (T-V)}{\partial q_j}\right) = 0$$

$\boxed{L = T - V}$ $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$ q.e.d.

⊛ Ejemplo de aplicación: m, $\underbrace{F(x)}_{\text{conservativa}}$

(a) Newton: $m\ddot{x} = F(x)$

$$m\ddot{x} = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} = F(x)$$

(Fuerza conservativa 1-D) $\boxed{F(x) = -\frac{dV}{dx}}$

$$m\int v\,dv = \int F(x)\,dx$$

$$\frac{1}{2}mv^2 = -V + C \rightarrow \boxed{\frac{1}{2}mv^2 + V = C = E}$$

(b) Lagrange — 1 partícula en 1D, sin ligaduras

$n = 1$ g.l. $\longrightarrow$ 1 coordenada generalizada $\boxed{q_1 = x}$

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \left(-\int F(x)dx\right) = \frac{1}{2} m \dot{x}^2 + \int F(x)dx$$

$$\boxed{\text{Ecuación de Lagrange} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$ (Fuerzas conservativas)

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(\int F(x)dx\right) = \int \frac{\partial F(x)}{\partial x} dx = \int \frac{dF}{dx} dx = \int dF = F(x)$$

$$\underbrace{\frac{\partial L}{\partial \dot{x}} = m\dot{x} \, , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}}$$

$$m\ddot{x} - F(x) = 0 \longrightarrow \boxed{F(x) = m\ddot{x}}$$

(c)

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_x \quad \text{Ecuación del movimiento (sean o no conservativas las fuerzas)}}$$

$$\underbrace{T = \frac{1}{2} m \dot{x}^2 \qquad \frac{\partial T}{\partial x} = 0 \qquad \frac{\partial T}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}}$$

$$Q_x = m\ddot{x} \longrightarrow \boxed{F(x) = m\ddot{x}}$$

$$\boxed{Q_j = \sum_{i=1}^{N} \vec{F}_i^{(e)} \frac{\partial \vec{r}_i}{\partial q_j} \qquad j = 1, \dots, 3N-k} \longrightarrow Q_x = F(x)\,\hat{\imath}\,\frac{\partial (x\hat{\imath})}{\partial x} = F(x)$$

$$\vec{r} = x\hat{\imath}$$

$$q_j = x$$

→ Momento generalizado y coordenada cíclica:

⊛ Ejemplo: L de una partícula libre en 1D
(sobre la que no actúa ninguna fuerza)

$$L = T - V = T = \frac{1}{2} m \dot{x}^2$$

$$\frac{\partial L}{\partial x} = 0 \,, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x \longrightarrow \text{momento lineal} \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(p_x) = 0 \rightarrow \boxed{p_x = c}$$

→ **Def.** – Momento generalizado o momento canónico conjugado de q_i

$$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$$

→ **Def.** – Coordenada cíclica:

Si $L(\dot{q}_i, q_i)$ no depende explícitamente de una o varias de las coordenadas generalizadas q_i, entonces q_i se llama coordenada cíclica.

Ec. de Lagrange asociada a la coordenada cíclica →

$$\boxed{\frac{\partial L}{\partial q_i} = 0} \qquad \frac{\partial L}{\partial \dot{q}_i} = p_i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \,, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\frac{d}{dt}(p_i) = 0$$

$$\Downarrow$$

$$\boxed{p_i = cte}$$

→ El momento generalizado asociado a una coordenada cíclica es invariante (se conserva, es una constante del movimiento)

↳ integral del movimiento

→ Energía cinética en coordenadas generalizadas:

Sea un sistema de N partículas $i = 1, 2, \dots, N \longrightarrow \vec{r}_i$

y n grados de libertad $\longrightarrow$ coordenadas q_j $\quad j = 1, 2, \dots, N$

$$\boxed{T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_i^2} \qquad \vec{r}_i = \vec{r}_i(q_j, t)$$

$$\dot{\vec{r}}_i = \sum_{j=1}^{n} \left(\frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right)$$

$$\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = (\dot{\vec{r}}_i)^2 = \dot{r}_i^2 = \left(\sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right) \left(\sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right) =$$

$$= \underbrace{\sum_{j,k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \frac{\partial \vec{r}_i}{\partial t} + 2 \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial t} \cdot \dot{q}_j}_{\dot{r}_i^2}$$

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_i^2$$

$$\left[\begin{aligned} a_{jk} &= \frac{1}{2} \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \frac{\partial \vec{r}_i}{\partial q_k} \\ b_j &= \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \frac{\partial \vec{r}_i}{\partial t} \\ c &= \frac{1}{2} \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial t} \frac{\partial \vec{r}_i}{\partial t} \end{aligned} \right] \qquad \boxed{T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c}$$

→ Sistema esclerónomo: $\boxed{\dfrac{\partial \vec{r}_i}{\partial t} = 0 \;\rightsquigarrow\; \vec{r}_i = \vec{r}_i(q_j)}$

$$\Downarrow \quad b_j = c = 0$$

$$\boxed{T = \sum_{j,k}^{n} a_{jk} \cdot \dot{q}_j \, \dot{q}_k}$$ Función homogénea cuadrática de las $\dot{q}_j$

⊛ Nota: $\boxed{f(x) \text{ homogénea de grado } n \equiv f(\lambda x) = \lambda^n f(x)}$

p.e. $f(x) = x^2$, $f(\lambda x) = \lambda^2 x^2 = \lambda^2 f(x) \to f(x)$ homogénea grado 2.

$$\frac{\partial T}{\partial \dot{q}_\ell} = \sum_{j,k}^{n} \left(a_{jk} \, \dot{q}_j \frac{\partial \dot{q}_k}{\partial \dot{q}_\ell} + a_{jk} \, \dot{q}_k \frac{\partial \dot{q}_j}{\partial \dot{q}_\ell} \right) =$$

$$= \sum_j a_{j\ell} \cdot \dot{q}_j + \sum_k a_{\ell k} \cdot \dot{q}_k$$

⊛ $\dfrac{\partial \dot{q}_k}{\partial \dot{q}_\ell} = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$

p.e. x, θ

$\dfrac{\partial \dot{x}}{\partial \dot{\theta}} = 0$, $\dfrac{\partial \dot{x}}{\partial \dot{x}} = 1$

$$\sum_{\ell=1}^{n} \frac{\partial T}{\partial \dot{q}_\ell} \cdot \dot{q}_\ell = \sum_{\ell,k} a_{\ell k} \cdot \dot{q}_k \dot{q}_\ell + \sum_{\ell,j} a_{j\ell} \, \dot{q}_j \dot{q}_\ell =$$

$$= 2 \sum_{j,k} a_{jk} \, \dot{q}_j \, \dot{q}_k = \boxed{2T}$$

Th. Euler: Si $f(\lambda y_i) = \lambda^n f(y_i) \Rightarrow \sum_i y_i \dfrac{\partial f}{\partial y_i} = n f$

($f(x)$ homogénea grado n)

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→ Simetría: transformación de coordenadas que deja invariable el sistema. Puede ser equivalente a una ley de conservación.

4 tipos:
- Homogeneidad del espacio → Momento lineal
- Homogeneidad del tiempo → Energía
- Isotropía del espacio → Momento angular
- Isotropía espacio-temporal → SPIN

(Cantidad conservada)

→ Invariancia temporal de L: $\boxed{\dfrac{\partial L}{\partial t} = 0}$ $\quad L(q_j, \dot{q}_j, t)$

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}$$

⊛ Ec. Lagrange: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right) - \dfrac{\partial L}{\partial q_j} = 0$, $\boxed{\dfrac{\partial L}{\partial q_j} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right)}$

$$\frac{dL}{dt} = \underbrace{\sum_{j=1}^{n} \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j}_{} + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} - \sum_{j=1}^{n} \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(L - \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial t}$$

⊛ Def.- Hamiltoniano (H):

$$\boxed{H \equiv \sum_{j=1}^{n} \dot{q}_j \left(\frac{\partial L}{\partial \dot{q}_j}\right) - L = \sum_{j=1}^{n} \dot{q}_j\, p_j - L}$$

$$\frac{d}{dt}\Big(\underbrace{L - \sum_{j=1}^{n} \dot{q}_j\, p_j}_{-H}\Big) = \frac{\partial L}{\partial t} \implies \boxed{-\frac{dH}{dt} = \frac{\partial L}{\partial t}}$$

→ Si L es invariante temporal, entonces el hamiltoniano se conserva:

$$\boxed{\frac{\partial L}{\partial t} = 0 \implies \frac{dH}{dt} = 0 \implies H \text{ se conserva } (H = cte)}$$

→ Si $L = T - V$, $\boxed{(a)}$ $V = V(q_j)$, $V \neq V(\dot{q}_j)$ → el potencial no depende de las velocidades generalizadas (F conservativas)

Si además $\underbrace{\vec{r}_i = \vec{r}_i(q_j)}$ $\boxed{(b)}$ ⟶ sist. escleronomo (no dependen del tiempo)

Entonces $\boxed{H = E}$

Dem: $$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \overset{(a)}{\cancel{\frac{\partial V}{\partial \dot{q}_j}}}^{\,0} = \frac{\partial T}{\partial \dot{q}_j} \quad (1)$$

$$\boxed{-H} = L - \sum_{j=1}^{n} \dot{q}_j\, p_j = T - V - \sum_{j=1}^{n} \dot{q}_j\, p_j =$$

$$= T - V - \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \overset{(1)}{=} T - V - \sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} =$$

$$\overset{(b)}{=} T - V - 2T = -T - V = -(T+V) = \boxed{-E}$$

S.P.6 sistema escleronomo ⇓ $\boxed{\sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T}$

$\boxed{H = E}$ q.e.d.

Scanned by CamScanner

(*) Si el sistema no es esclerónomo ($\vec{r}_i = \vec{r}_i(q_j, t)$)

$$\boxed{H \neq E}$$

- Casos posibles:

  ① $H \neq E$ y no se conserva ninguno.

  ② $H \neq E$ se conserva H o se conserva E.

  ③ $H = E$ se conservan los dos o no se conserva ninguno.

→ Simetrías de L y leyes de conservación:

$$\frac{\partial L}{\partial t} = 0 \Rightarrow H = cte$$

- Invariancia temporal $\frac{dL}{dt} = 0 \Rightarrow H = cte$

- Invariancia traslacional y conservación del momento lineal:

  Sea q_j tal que dq_j es una traslación espacial:

$$q_j \equiv x, y, z, r, \theta, \ldots$$

Claramente $\frac{\partial T}{\partial q_j} = 0$

Ec. Lagrange: $\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)}_{(a)} = \underbrace{\frac{\partial L}{\partial q_j}}_{(b)}$

$$\text{(a)} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) =$$

$$\underset{(\text{F conservativas})}{\overset{V \neq V(\dot{q}_j)\ ⊛}{=}} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{d}{dt}(p_j) = \dot{p}_j$$

$$\text{(b)} \quad \frac{\partial L}{\partial q_j} = \frac{\partial (T-V)}{\partial q_j} = \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \underset{\frac{\partial T}{\partial q_j}=0}{=} -\frac{\partial V}{\partial q_j}$$

(a) = (b)

$$\boxed{\dot{p}_j = -\frac{\partial V}{\partial q_j}}$$

$$\Downarrow$$

Fuerzas conservativas $\Longrightarrow$ $\boxed{Q_j = -\frac{\partial V}{\partial q_j}}$ $\Longrightarrow$ $\boxed{\dot{p}_j = Q_j}$

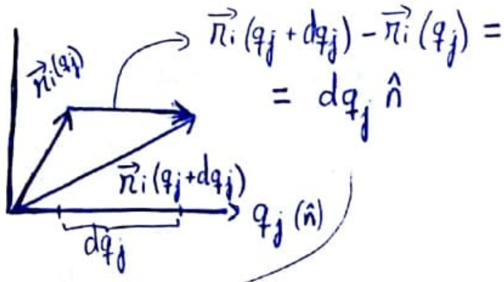
"2ª Ley de Newton Generalizada"

Demostremos:

1. Q_j es la componente de la $\vec{F}$ en la dirección de q_j.
2. p_j es el momento lineal a lo largo de q_j.

① $$Q_j \equiv \sum_{i=1}^{N} \vec{F}_i^{(e)} \frac{\partial \vec{r}_i}{\partial q_j}$$

(Fuerza generalizada)



$$\frac{\partial \vec{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\vec{r}_i(q_j+dq_j) - \vec{r}_i(q_j)}{dq_j} = \lim_{dq_j \to 0} \frac{dq_j\,\hat{n}}{dq_j} = \hat{n}$$

$$Q_j = \sum_{i=1}^{N} \vec{F}_i^{(e)} \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} \vec{F}_i^{(e)} \cdot \hat{n} = \boxed{\hat{n} \sum_{i=1}^{N} \vec{F}_i^{(e)}}$$

proyección de la fuerza total en la dirección de $\hat{n}$ (dirección de q_j) q.e.d.

(2) $T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2}$

$$(V \neq V(\dot q_j))$$

$$p_j = \frac{\partial L}{\partial \dot q_j} = \frac{\partial (T-V)}{\partial \dot q_j} = \frac{\partial T}{\partial \dot q_j} - \overbrace{\frac{\partial V}{\partial \dot q_j}}^{0} = \frac{\partial T}{\partial \dot q_j} = \frac{1}{2} \sum_{i=1}^{N} m_i \, 2 \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot q_j} =$$

$$= \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot q_j}$$

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot q_j} = ? \qquad \dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{dq_k}{dt} + \frac{\partial \vec{r}_i}{\partial t} \frac{dt}{dt} =$$

$\vec{r}_i(q_k, t)$, $k = 1,\dots,n$

$$= \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k} \cdot \dot q_k + \frac{\partial \vec{r}_i}{\partial t}$$

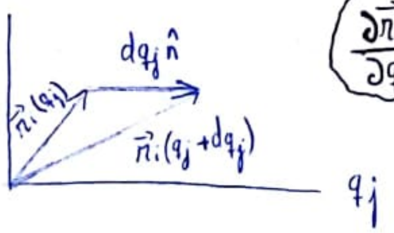
$\vec{r}_i \neq \vec{r}_i(\dot q_j)$ (*)

$$\boxed{\frac{\partial \dot{\vec{r}}_i}{\partial \dot q_j}} = \sum_k \overbrace{\frac{\partial^2 \vec{r}_i}{\partial \dot q_j \partial q_k}}^{0} \cdot \dot q_k + \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \frac{\partial \dot q_k}{\partial \dot q_j} + \overbrace{\frac{\partial^2 \vec{r}_i}{\partial \dot q_j \partial t}}^{0} =$$

$$= \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{\partial \dot q_k}{\partial \dot q_j} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \underbrace{\delta_{jk}}_{\text{Delta de Kronecker}} = \boxed{\frac{\partial \vec{r}_i}{\partial q_j}}$$

$$\boxed{p_j} = \sum_{i=1}^{N} m_i \vec{v}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot q_j} = \sum_{i=1}^{N} m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} m_i \vec{v}_i \cdot \hat{n} =$$

$$= \hat{n} \cdot \sum_{i=1}^{N} \vec{v}_i \cdot m_i \equiv \text{proyección del momento lineal a lo largo de } q_j$$



$$\frac{\partial \vec{r}_i}{\partial q_j} = \hat{n}$$

$$\boxed{\dot p_j \equiv Q_j}$$

→ Si q_j es cíclica: $\left\{ \begin{array}{l} \dfrac{\partial L}{\partial q_j} = 0 \\[2ex] \dfrac{\partial T}{\partial q_j} = 0 \end{array} \right.$

$$\dot p_j = 0 \Longleftarrow Q_j \underset{\uparrow}{=} -\frac{\partial V}{\partial q_j} = 0 \longleftarrow \text{porque}$$

F. conservativas

$\boxed{p_j = \text{cte}}$

Teorema:

Si una coordenada cíclica q_j es tal que dq_j corresponde a una traslación espacial, entonces se conserva la componente del momento lineal en la dirección q_j $(\hat{n})$.

□ Invariancia rotacional de L y conservación del momento angular:

Teorema:

Si una coordenada cíclica q_j es tal que dq_j corresponde a una rotación del sistema, entonces se conserva la componente del momento angular en torno al eje de rotación de q_j $(\hat{n})$

Dem:

$$\left.\begin{array}{l} \dfrac{\partial T}{\partial q_j} = 0 \\ V \neq V(\dot{q}_j) \end{array}\right\} \quad \boxed{\dot{p}_j = Q_j}$$

① Q_j es la componente del torque alrededor del eje de rotación de q_j

$$Q_j = \hat{n} \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{(e)}$$

(2) p_j es la componente del momento angular alrededor del eje de rotación de q_j:

$$p_j = \hat{n} \cdot \sum_{i=1}^{N} (\vec{r}_i \times m_i \vec{v}_i) = \hat{n} \cdot \vec{L}$$

$\longrightarrow$ Si q_j es cíclica $\longrightarrow$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial q_j} = 0 \\ \dfrac{\partial T}{\partial q_j} = 0 \end{array}\right\} \Rightarrow \frac{\partial V}{\partial q_j} = 0$$

$$\Downarrow$$

$$\boxed{Q_j = -\frac{\partial V}{\partial q_j} = 0}$$

$$\Leftarrow \quad \dot{p}_j = 0$$

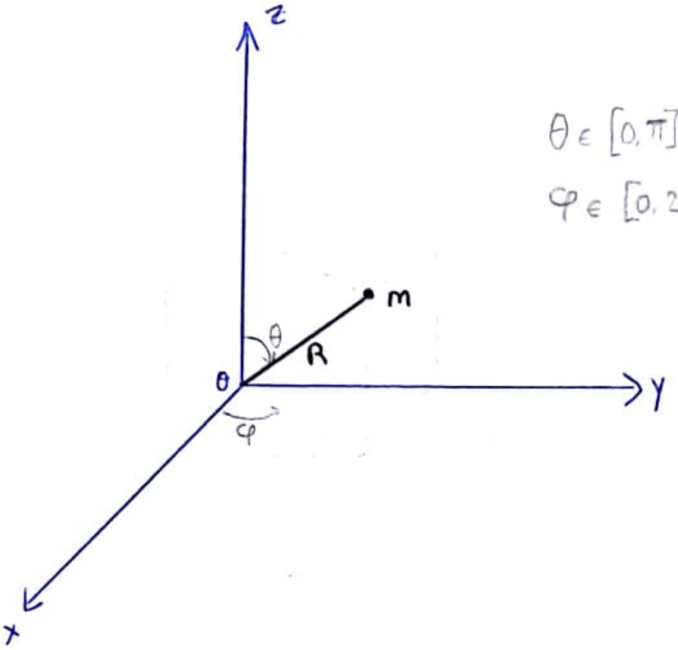
$$\Downarrow$$

$$\boxed{p_j = cte} \longrightarrow$$ Se conserva la componente del momento angular.

⊛ Teorema de Noether:

Cualquier simetría diferenciable en un sistema físico tiene su correspondiente ley de conservación.

Péndulo esférico (Symon 380-5)



$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi]$

1 partícula en 3D $\to N = 3$

$\underbrace{x^2 + y^2 + z^2 = R^2}_{1 \text{ ligadura}}$

$n = 3 - 1 = 2$ g. l.

2 coordenadas generalizadas

$q_1 = \theta \quad q_2 = \varphi$

$$\begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R\cos\theta \end{cases}$$

$$\begin{cases} \dot{x} = R\cos\varphi\cos\theta\,\dot{\theta} - R\sin\theta\sin\varphi\,\dot{\varphi} \\ \dot{y} = R\sin\varphi\cos\theta\,\dot{\theta} + R\sin\theta\cos\varphi\,\dot{\varphi} \\ \dot{z} = -R\sin\theta\,\dot{\theta} \end{cases}$$

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) =$$

$$= \frac{1}{2} m\Big[R^2\cos^2\varphi\cos^2\theta\,\dot{\theta}^2 + R^2\sin^2\varphi\sin^2\theta\,\dot{\varphi}^2 -$$

$$\cancel{-2R^2\dot{\theta}\dot{\varphi}\cos\varphi\sin\varphi\cos\theta\sin\theta} + R^2\sin^2\varphi\cos^2\theta\,\dot{\theta}^2 +$$

$$+ R^2\cos^2\varphi\sin^2\theta\,\dot{\varphi}^2 + \cancel{2R^2\dot{\theta}\dot{\varphi}\sin\varphi\cos\varphi\cos\theta\sin\theta} +$$

$$+ R^2\sin^2\theta\,\dot{\theta}^2\Big] =$$

$$= \frac{1}{2} m\Big[R^2\dot{\theta}^2\cos^2\theta\,\overbrace{[\cos^2\varphi + \sin^2\varphi]}^{1} + R^2\sin^2\theta\,\dot{\varphi}^2\,\overbrace{[\cos^2\varphi + \sin^2\varphi]}^{1} + R^2\sin^2\theta\,\dot{\theta}^2\Big] =$$

$$= \frac{1}{2} m\left[R^2\dot{\theta}^2 + R^2\dot{\varphi}^2\sin^2\theta\right]$$

$$\boxed{T = \frac{1}{2} mR^2\left[\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right]}$$

(*) $v^2 = v_\theta^2 + v_\varphi^2$

$v_\theta = R\dot{\theta} \qquad v_\varphi = R\dot{\varphi}\sin\theta$

(*) $V(z=0) = 0 \longrightarrow V = mgz = mgR\cos\theta$

$L = T - V \Rightarrow \boxed{L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\varphi}^2\sin^2\theta - mgR\cos\theta}$

Ecs. de Lagrange: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right) - \dfrac{\partial L}{\partial q_j} = 0$

(θ) $\dfrac{\partial L}{\partial \theta} = mgR\sin\theta + mR^2 \sin\theta\cos\theta\,\dot{\varphi}^2$

$\dfrac{\partial L}{\partial \dot\theta} = mR^2\dot\theta \qquad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\theta}\right) = mR^2\ddot\theta$

$$mR^2\ddot\theta = mR^2\sin\theta\cos\theta\,\dot\varphi^2 + mgR\sin\theta$$

$$\boxed{\ddot\theta = \sin\theta\cos\theta\,\dot\varphi^2 + \frac{g}{R}\sin\theta} \quad (1)$$

⊛ Si $\dot\varphi = 0 \Rightarrow$ péndulo simple (plano): $\boxed{\ddot\theta = \frac{g}{R}\sin\theta}$

(φ) $\dfrac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi$ cíclica $\Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\varphi}\right) = 0$

$\Downarrow$

$$\boxed{p_\varphi = \frac{\partial L}{\partial \dot\varphi} = cte}$$

se conserva la componente del momento angular alrededor del eje z

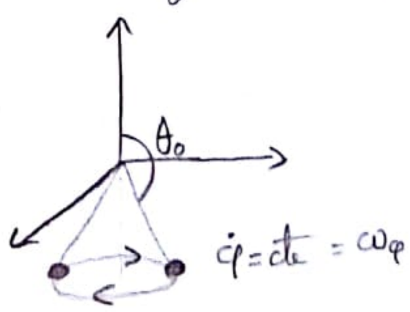
$$\boxed{p_\varphi} = \frac{\partial L}{\partial \dot\varphi} = \boxed{mR^2\sin^2\theta\,\dot\varphi = cte} \quad (2)$$

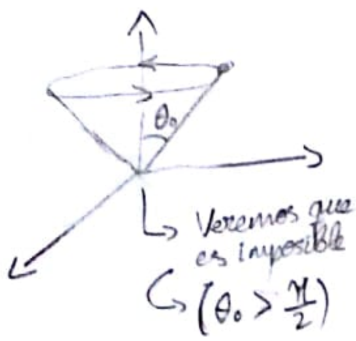
$\Downarrow$

⊛ Fijado el ángulo θ también queda fijada la velocidad angular $\dot\varphi$ (y viceversa)

⊛ p.e. $\sin^2\theta = \sin^2\theta_0$

↳ $\dot\varphi = cte = \dot\varphi_0$





$(2) \Longrightarrow \dot{\varphi} = \dfrac{P_\varphi}{mR^2 \sin^2\theta} \longrightarrow (1) \quad \ddot{\theta} = \sin\theta\cos\theta \dfrac{P_\varphi^2}{m^2R^4\sin^4\theta} + \dfrac{g}{R}\sin\theta$

$$\Downarrow$$

$$\boxed{\ddot{\theta} = \frac{P_\varphi^2 \cos\theta}{m^2 R^4 \sin^3\theta} + \frac{g}{R}\sin\theta}$$

→ Ecuación desacoplada en θ.

→ Energía y Hamiltoniano:

$\dfrac{\partial L}{\partial t} = 0 \Rightarrow \boxed{H = cte} \longrightarrow$ El Hamiltoniano se conserva

(i) $\vec{r}(x, y, z)$ no depende explícitamente de t

(ii) V no depende de las velocidades generalizadas

$\Rightarrow \boxed{H = E}$

$\Downarrow$

La energía también se conserva (como era de esperar, fuerzas conservativas)

$$H = p_\theta \dot{\theta} + p_\varphi \dot{\varphi} - L =$$

$$= mR^2\dot{\theta}^2 + mR^2\sin^2\theta\,\dot{\varphi}^2 - \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}mR^2\sin^2\theta\,\dot{\varphi}^2 + mgR\cos\theta =$$

$$= \underbrace{\frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\sin^2\theta\,\dot{\varphi}^2}_{T} + \underbrace{mgR\cos\theta}_{V}$$

$$E = T + V$$

$$\hookrightarrow E = \underbrace{\frac{1}{2}mR^2\dot{\theta}^2}_{\text{Término cinético } (\geq 0)} + \underbrace{\frac{P_\varphi^2}{2mR^2\sin^2\theta} + mgR\cos\theta}_{V_{efectivo}(\theta)}$$

$V_{efectivo}(\theta) \longleftarrow$ Método de Routh

Solo están permitidos los $\theta \,/\, E \geq V_{ef}(\theta)$

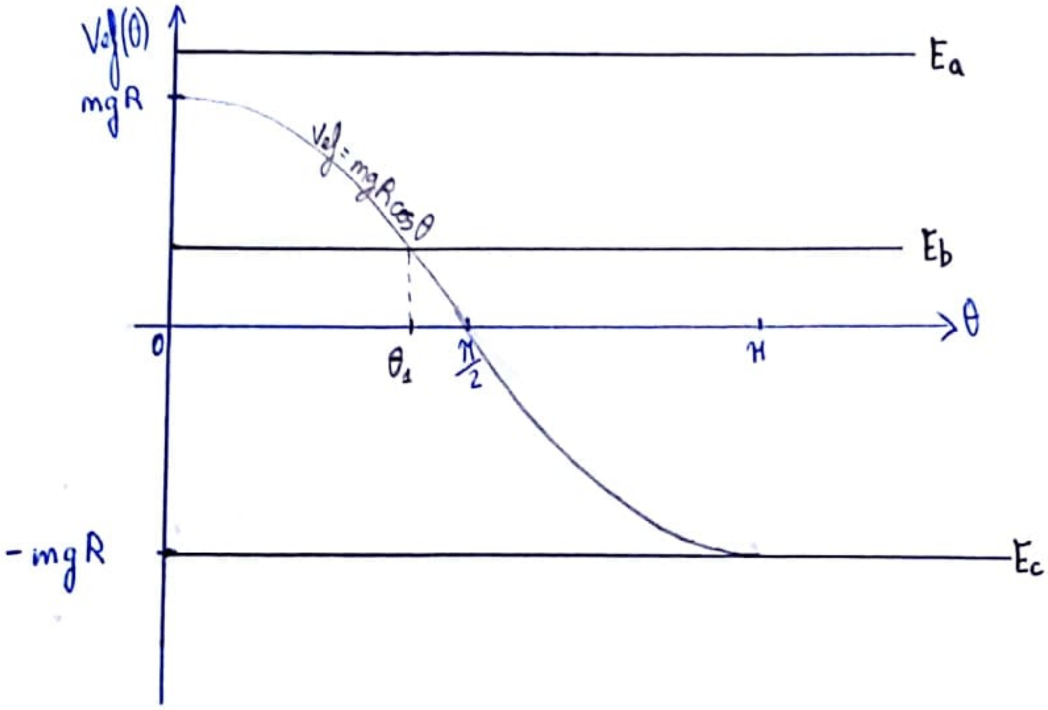
⊛ $\boxed{\text{Caso I}}$: $\underbrace{P_\varphi = 0}$ $\Rightarrow$ $\underbrace{\dot{\varphi} = 0}$ No hay rotación en torno al eje Z

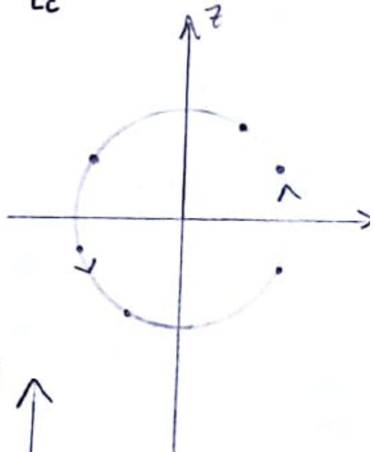
$\Downarrow$ $\qquad\qquad$ $\downarrow$

$\varphi = \varphi_0$ ⊛ Supondremos $\boxed{\varphi_0 = \pm \frac{\pi}{2}}$

m se mueve en el plano YZ

$$\boxed{V_{ef}(\theta) = mgR\cos\theta}$$





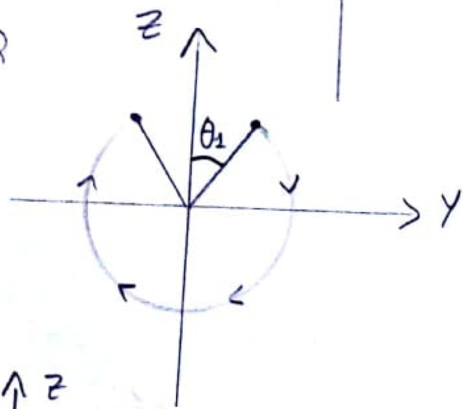
(Ia) $E = E_a \geq mgR \Longrightarrow \theta \in [0, \pi]$

$(\varphi_0 = \pm \frac{\pi}{2})$

(Ib) $E = E_b$, $-mgR < E_b < mgR$

$\theta \in [\theta_1, \pi]$

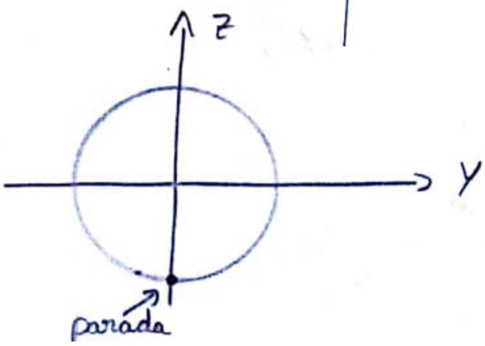
$(\varphi_0 = \pm \frac{\pi}{2})$



(Ic) $E = E_c = -mgR$

$\theta = \pi$

$(\varphi_0 = \pm \frac{\pi}{2})$



$$\frac{dV_{ef}(\theta)}{d\theta} = -mgR\sin\theta \quad , \quad -mgR\sin\theta = 0$$

$$\hookrightarrow \boxed{\theta = 0, \pi} \rightarrow \text{Candidatos a extremo relativo}$$

$$\frac{d^2V_{ef}}{d\theta^2} = -mgR\cos\theta$$

$$\left.\frac{d^2V_{ef}}{d\theta^2}\right|_{\theta=0} = -mgR < 0 \Rightarrow \theta = 0 \rightarrow V_{ef}(0) \text{ máximo del potencial (punto de equilibrio inestable)}$$

$$\left.\frac{d^2V_{ef}}{d\theta^2}\right|_{\theta=\pi} = mgR > 0 \Rightarrow \theta = \pi \text{ , } V_{ef}(\pi) \text{ mínimo del potencial (punto de equilibrio estable)}$$

⊛ **Caso II :** $\quad P_\varphi \neq 0 \Rightarrow \dot{\varphi} \neq 0$

$$\boxed{V_{ef}(\theta) = \frac{P_\varphi^2}{2mR^2\sin^2\theta} + mgR\cos\theta}$$

→ Mínimos ? $\longrightarrow \theta_0$

$$\frac{dV_{ef}}{d\theta} = \frac{P_\varphi^2}{2mR^2}\,\frac{-2\sin\theta\cos\theta}{\sin^4\theta} - mgR\sin\theta = -\frac{P_\varphi^2}{mR^2}\,\frac{\cos\theta}{\sin^3\theta} - mgR\sin\theta$$

$$\left.\frac{dV_{ef}}{d\theta}\right|_{\theta_0} = 0 \quad , \quad \boxed{\underbrace{mgR}_{>0} = -\underbrace{\frac{P_\varphi^2}{mR^2}}_{>0}\,\frac{\cos\theta_0}{\sin^4\theta_0}\}\geq 0} \Rightarrow \underbrace{\cos\theta_0 < 0}_{\boxed{\theta_0 > \frac{\pi}{2}}}$$

⊛ Aproximación para pequeñas oscilaciones en torno al mínimo:

$$\frac{d^2V_{ef}}{d\theta^2} = -\frac{P_\varphi^2}{mR^2}\,\frac{-\sin\theta}{\sin^3\theta} - \frac{P_\varphi^2}{mR^2}\,\frac{-\cos\theta\, 3\sin^2\theta\cos\theta}{\sin^6\theta} - mgR\cos\theta =$$

$$= -\frac{P_\varphi^2}{mR^2}\,\frac{-\sin^2\theta - 3\cos^2\theta}{\sin^4\theta} - mgR\cos\theta =$$

$$= \frac{P_\varphi^2}{mR^2}\,\frac{1 + 2\cos^2\theta}{\sin^4\theta} - mgR\cos\theta$$

$$\left.\frac{d^2V_{ef}}{d\theta^2}\right|_{\theta_0} = \underbrace{\frac{P_\varphi^2}{mR^2}\,\frac{1+2\cos^2\theta_0}{\sin^4\theta_0}}_{>0} \underbrace{- mgR\underbrace{\cos\theta_0}_{<0}}_{>0} > 0 \Rightarrow V_{ef}(\theta_0) \text{ mínimo } ✓$$

Desarrollo Taylor orden 2 en torno al mínimo:

$$V_{ef}(\theta) = V_{ef}(\theta_0) + \cancel{\left.\frac{dV_{ef}}{d\theta}\right|_{\theta_0}}^{\,0}(\theta-\theta_0) + \frac{1}{2}\left.\frac{d^2V_{ef}}{d\theta^2}\right|_{\theta_0}(\theta-\theta_0)^2 + \dots \simeq$$

$$\simeq V_{ef}(\theta_0) + \frac{1}{2}k(\theta-\theta_0)^2$$

$$k = \left.\frac{d^2V_{ef}}{d\theta^2}\right|_{\theta_0} = \frac{P_\varphi^2}{mR^2}\,\frac{1+2\cos^2\theta_0}{\sin^4\theta_0} - mgR\cos\theta_0$$

$$\leadsto \left.\frac{dV_{ef}}{d\theta}\right|_{\theta_0} = 0 \equiv -\frac{P_\varphi^2}{mR^2}\,\frac{\cos\theta_0}{\sin^4\theta_0} - mgR = 0$$

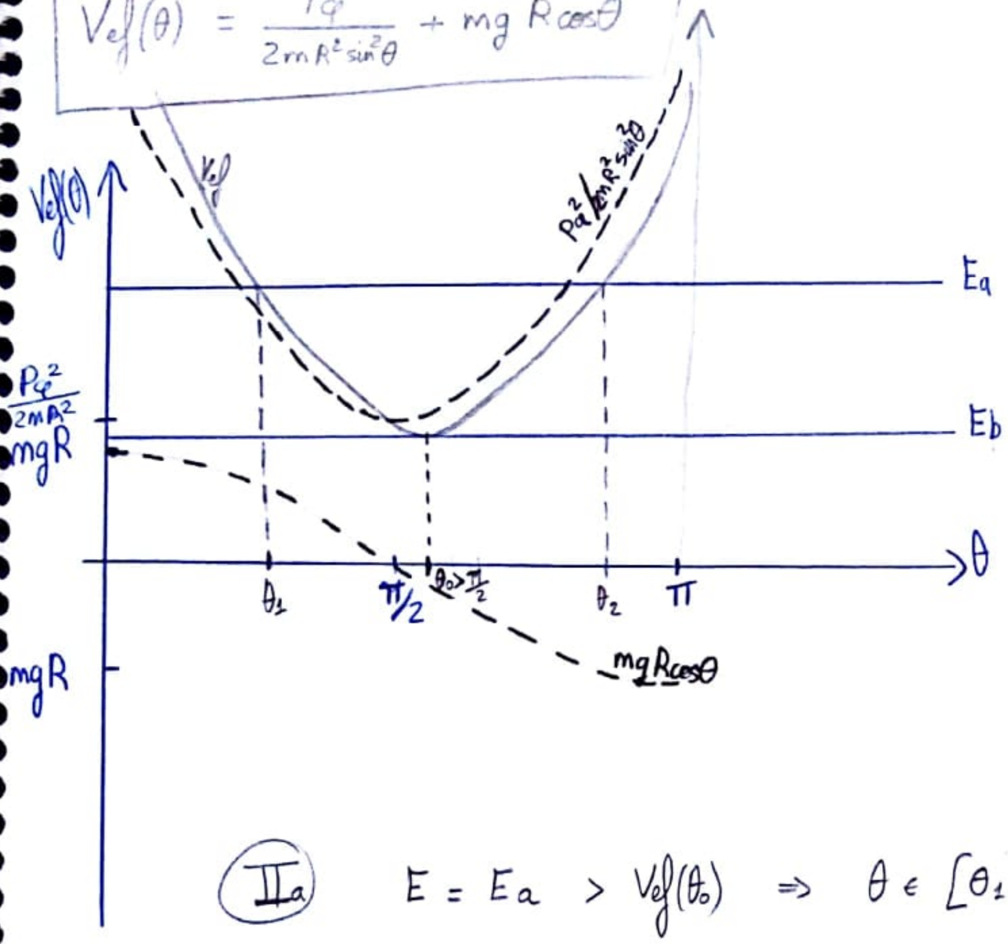
$$\boxed{\frac{P_\varphi^2}{mR^2} = -\frac{mgR\sin^4\theta_0}{\cos\theta_0}}$$

$$\boxed{k} = -mgR\,\frac{1+2\cos^2\theta_0}{\cos\theta_0} - mgR\cos\theta_0$$

$$= \frac{-mgR(1+2\cos^2\theta_0+\cos^2\theta_0)}{\cos\theta_0} = \boxed{\frac{-mgR}{\cos\theta_0}(1+3\cos^2\theta_0)}$$

$$\cos\theta_0 < 0 \Longrightarrow k > 0 \ ✓ \longrightarrow \text{oscilador } \boxed{\omega = \sqrt{\frac{k}{m}}}$$

$$\boxed{V_{ef}(\theta) = \frac{P_\varphi^2}{2mR^2\sin^2\theta} + mgR\cos\theta}$$



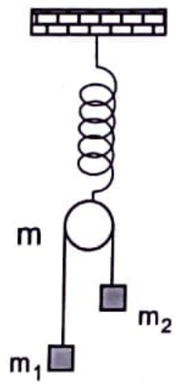
(IIa) $E = E_a > V_{ef}(\theta_0) \Rightarrow \theta \in [\theta_1, \theta_2]$

(IIb) $E = E_b = V_{ef}(\theta_0) \Rightarrow \theta = \theta_0$ (circular)

Vimos que $\boxed{\theta_0 > \frac{\pi}{2}}$

2. Mecánica Analítica

2.1. Utilizando o principio de D'Alembert, atopar as condicións de equilibrio para o sistema da figura.



2.2. Unha partícula esvara sen rozamento por un arame con forma de cicloide de ecuacións $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$. Atopar a función de Lagrange e as ecuacións do movemento.

2.3. a) Un péndulo oscilante consiste nunha masa m pendurada dun resorte de constante k e lonxitude en repouso l_0 que pode moverse no plano vertical. Obter as ecuacións diferenciais do movemento. b) Atopar as ecuacións diferenciais do movemento dun péndulo doble.

2.4. Unha partícula de masa m está limitada a moverse sobre unha circunferencia horizontal de radio a. Inicialmente, a partícula ten velocidade v_0. O movemento está sometido a unha resistencia do aire proporcional ao cadrado da velocidade. Usando o método de Lagrange, obter a posición da partícula en función do tempo.

2.5. Estudar o movemento dun péndulo de masa m e lonxitude l pendurado dunha masa M que se move sen rozamento sobre unha recta horizontal.

2.6. Dous puntos materiais de masa m_1 e m_2 están unidos por un fío que pasa por un burato practicado nunha mesa lisa, de xeito que m_1 descansa sobre a superficie da mesa e m_2 pendura do fío. Asumindo que m_2 só se move ao longo dunha recta vertical, escribir as ecuacións diferenciais do movemento e estudar o seu significado físico. Reducir o problema a unha soa ecuación diferencial de segunda orde, obter unha integral primeira da ecuación e discutir o seu significado físico.

2.7. Estudar o movemento dunha partícula que se move sobre a superficie dun cilindro vertical de radio R sometida a unha forza $\vec{F} = -k\vec{r}$.

2.8. Estudar o movemento dunha partícula de masa m limitada a moverse sobre a superficie dun cono vertical invertido de semiángulo α e sometida á forza gravidade.

2.9. Supóñase un péndulo simple plano composto dunha masa m suxeita a unha corda de lonxitude l. Unha vez que o péndulo comenzou a moverse, a lonxitude da corda vaise acurtando a velocidade constante:

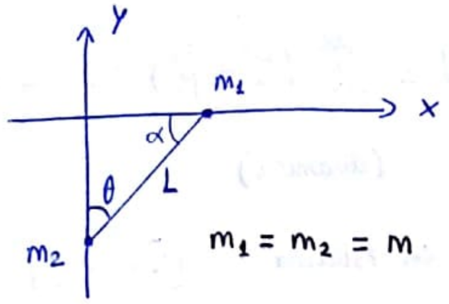
$$\frac{dl}{dt} = -\alpha = cte.$$

quedando fixo o punto de suspensión. Determinar a función de Lagrange e a función de Hamilton e discutir a conservación desta última e a enerxía.

2.10. Un corpo puntual de masa m esvara sen rozamento por unha recta AB partindo da orixe O. A recta móvese no plano XY con velocidade angular uniforme ω ao redor da orixe e en $t = 0$ coincide co eixo X. Integrar a ecuación do movemento se a partícula está sometida á acción da gravidade que actúa na dirección negativa do eixo Y.

2.11. Unha esfera pequena de masa m esvara sen rozamento sobre un aro circular de radio a. O aro atópase nun plano vertical e xira ao redor dun diámetro vertical con velocidade angular constante ω. Estudar a natureza do movemento.

Boletín : Lagrange



$\boxed{V=0}$

$m_1 = m_2 = m$

2D → 2 partículas

$\underbrace{\qquad\qquad}$

N = 4 coordenadas

3 Ligaduras : $y_1 = 0 \quad x_2 = 0$

$y_2^2 + x_1^2 = L^2$

g.l. ⟶ $\boxed{n=1}$ coordenada generalizada: $\boxed{q_1 = \theta}$

$$\begin{cases} x_1 = L\sin\theta & y_1 = 0 \\ x_2 = 0 & y_2 = -L\cos\theta \end{cases}$$

$$L = T - V$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \cancel{\dot{y}_1^2}^{\,0}) + \frac{1}{2} m_2 (\cancel{\dot{x}_2^2}^{\,0} + \dot{y}_2^2)$$

$$\dot{x}_1 = L\cos\theta\,\dot\theta \longrightarrow \dot{x}_1^2 = L^2\cos^2\theta \cdot \dot\theta^2$$

$$\dot{y}_2 = L\sin\theta\,\dot\theta \longrightarrow \dot{y}_2^2 = L^2\sin^2\theta\,\dot\theta^2$$

$\boxed{m = m_1 = m_2}$

$$T = \frac{1}{2} m_1 l^2\cos^2\theta\,\dot\theta^2 + \frac{1}{2} m_2\, l^2\sin^2\theta\,\dot\theta^2 = \frac{1}{2} m\, l^2\dot\theta^2(\cos^2\theta + \sin^2\theta) = \frac{m l^2\dot\theta^2}{2}$$

$$V = m_1 g \cancel{y_1}^{\,0} + m_2 g\, y_2 = -m g\, l\cos\theta$$

$$\boxed{L = T - V = \frac{m l^2\dot\theta^2}{2} + m g\, l\cos\theta}$$

Ecuación de Lagrange: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

$$\frac{\partial L}{\partial\theta} = -m g\, l\sin\theta \qquad \frac{\partial L}{\partial\dot\theta} = m l^2\dot\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = m l^2\ddot\theta$$

$$l^2 m\ddot\theta + m g\, l\sin\theta = 0 \longrightarrow l^2\ddot\theta + g\, l\sin\theta = 0$$

$$l\ddot\theta + g\sin\theta = 0$$

$$\boxed{\ddot\theta = -\frac{g}{l}\sin\theta}$$

péndulo simple

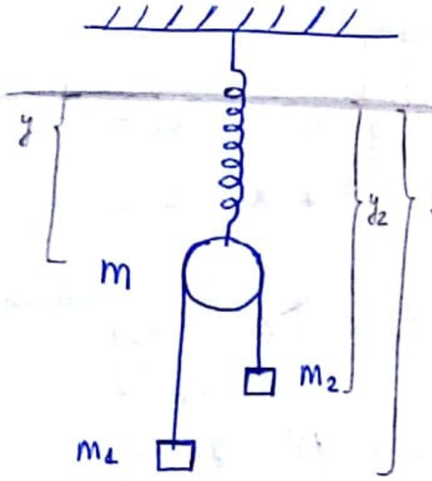
(2.1) Usar el principio de D'Alembert para encontrar las condiciones de equilibrio:

$$\delta W = \sum_{i=1}^{N} (\vec{F_i} - \dot{\vec{p_i}})\, \delta \vec{r_i} = 0$$

(dinámica)

$\rightarrow$ en estática: $\boxed{Q_j = 0}$

$$\boxed{\sum_{i=1}^{N} \vec{F_i}^{(e)} \cdot \frac{\partial \vec{r_i}}{\partial q_j} = 0}$$



Tomando como ref. el extremo del muelle en reposo

3 partículas 1D: $\boxed{N = 3 \text{ coordenadas}}$

1 Ligadura: $L = (y_1 - y) + (y_2 - y)$ $\quad \boxed{n = 3 - 1 = 2}$ g. l.

lonxitude da corda $\quad \boxed{L = y_1 + y_2 - 2y}$

2 g. l.

$\rightarrow$ 2 coordenadas generalizadas: $\boxed{q_1 = y}$ $\quad \boxed{q_2 = y_1 = z}$

$$\begin{cases} y = y \\ y_1 = z \\ y_2 = L + 2y - y_1 = L + 2y - z \end{cases}$$

Fuerzas externas: $\quad F = mg - ky, \quad F_1 = m_1 g, \quad F_2 = m_2 g$

$$Q_y = F \frac{\partial y}{\partial y} + F_1 \frac{\partial y_1}{\partial y}^{\,0} + F_2 \frac{\partial y_2}{\partial y} = (mg - ky) + 2 m_2 g$$

$$Q_z = F \cancel{\frac{\partial y}{\partial z}}^{\,0} + F_1 \frac{\partial y_1}{\partial z} + F_2 \frac{\partial y_2}{\partial z} = m_1 g - m_2 g$$

Ppio D'Alembert $\Rightarrow$ $\quad Q_y = (mg - ky) + 2 m_2 g = 0$

$\quad Q_z = m_1 g - m_2 g = 0, \quad m_1 - m_2 = 0,$

$\boxed{m_1 = m_2} = m'$

$$mg - ky + 2m'g = 0$$

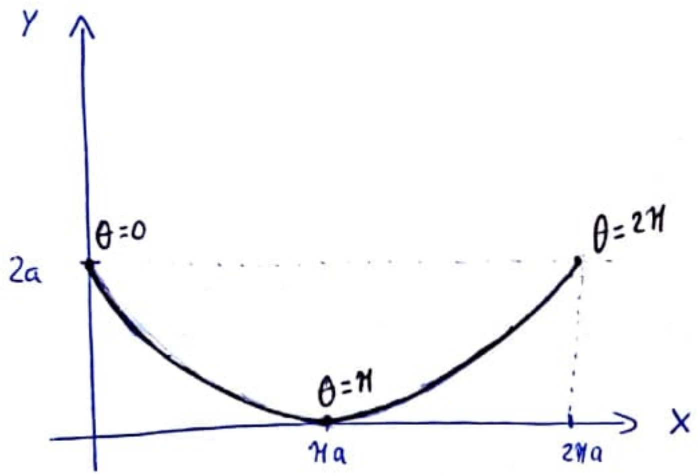
$$\boxed{y = \frac{(2m' + m)g}{k}}$$

Condiciones de equilibrio

② 1 partícula $\longrightarrow$ esvara sen rozamento por un cicloide

¿ L, ec. movemento ?

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1+\cos\theta) \end{cases}$$



$\theta = 0 \quad \begin{cases} x = 0 \\ y = 2a \end{cases}$

$\theta = \pi \quad \begin{cases} x = \pi a \\ y = 0 \end{cases}$

$\theta = 2\pi \quad \begin{cases} x = 2\pi a \\ y = 2a \end{cases}$

2D $\times$ 1 partícula $\longrightarrow$ $N = 2$

1 ligadura : $(x, y) \in$ cicloide $k = 1$

$\longrightarrow \boxed{n = 2 - 1 = 1 \text{ g. l.}}$

coordenada generalizada : $\boxed{q_1 = \theta}$

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1+\cos\theta) \end{cases}$$

$$\begin{cases} \dot{x} = a(\dot\theta - \cos\theta\,\dot\theta) = a\dot\theta(1-\cos\theta) \\ \dot{y} = a(-\cos\theta\,\dot\theta) = -a\dot\theta\sin\theta \end{cases}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) =$$

$$= \frac{1}{2} m \left[a^2\dot\theta^2(1-\cos\theta)^2 + a^2\dot\theta^2\sin^2\theta\right] =$$

$$= \frac{1}{2} m a^2 \dot\theta^2 \left[1 + \cos^2\theta - 2\cos\theta + \sin^2\theta\right] =$$

$$= m a^2 \dot\theta^2 (1 - \cos\theta)$$

$\boxed{y = 0} \longleftrightarrow \boxed{V = 0} \longrightarrow V = mgy = mga(1+\cos\theta)$

$$\boxed{L = T - V = ma^2\dot\theta^2(1-\cos\theta) - mga(1+\cos\theta)}$$

Ecuación de Lagrange:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0}$$

$$\frac{\partial L}{\partial \theta} = m a^2 \dot\theta^2 \sin\theta + m g a \sin\theta$$

$$\frac{\partial L}{\partial \dot\theta} = 2 m a^2 (1 - \cos\theta)\dot\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 2ma^2\left[\sin\theta\,\dot\theta^2 + (1-\cos\theta)\ddot\theta\right]$$

$$2 m a^2 \sin\theta\,\dot\theta^2 + 2ma^2(1-\cos\theta)\,\ddot\theta - ma^2\dot\theta^2\sin\theta - mga\sin\theta = 0$$

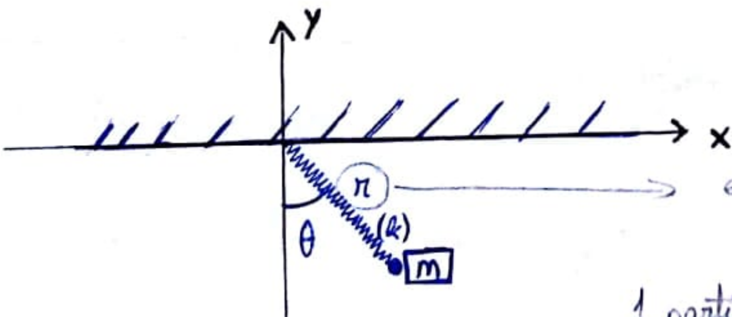
$$a \sin\theta\,\dot\theta^2 + 2a(1-\cos\theta)\,\ddot\theta - g\sin\theta = 0$$

$$2a(1-\cos\theta)\,\ddot\theta + a\sin\theta\,\dot\theta^2 = g\sin\theta$$

$$\boxed{\left(\frac{1-\cos\theta}{\sin\theta}\right)\ddot\theta + \frac{\dot\theta^2}{2} = \frac{g}{2a}} \longrightarrow \text{Ecuación del movimiento}$$

2.3 a) Péndulo oscilante: m pendurada dun resorte (k, l_0):

movemento no plano vertical



estiramento del muelle $\equiv r - l_0$

1 partícula 2D $\longrightarrow$ $N = 2$ (x, y)

0 Ligaduras $\longrightarrow$ $\boxed{n = 2 \text{ g. l.}}$

2 coordenadas generalizadas: $\boxed{q_1 = r}$ $\boxed{q_2 = \theta}$

$$\begin{cases} x = r\sin\theta \\ y = -r\cos\theta \end{cases}$$

Scanned by CamScanner

$\boxed{L = T - V}$

$$\begin{cases} \dot{x} = r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta \\ \dot{y} = -\dot{r}\cos\theta + r\sin\theta\dot{\theta} \end{cases}$$

$\boxed{2\,Ejs}$

$$\boxed{T} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) =$$

$$= \frac{1}{2}m\left[r^2\dot{\theta}^2\cos^2\theta + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2\dot{r}r\dot{\theta}\sin\theta\cos\theta\right] =$$

$$= \frac{1}{2}m\left[r^2\dot{\theta}^2 + \dot{r}^2\right](\sin^2\theta + \cos^2\theta) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m(v_r^2 + v_\theta^2)$$

$$V = -mgr\cos\theta + \frac{1}{2}k(r - l_0)^2 \xrightarrow{\circledast\ \tilde{r} = r - l_0} \boxed{V = -mgr\cos\theta + \frac{1}{2}k\tilde{r}^2}$$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k\tilde{r}^2}$$

→ Ecuaciones de Lagrange: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q} = 0}$

(r) $\frac{\partial L}{\partial r} = \frac{1}{2}m\,2r\dot{\theta}^2 + mg\cos\theta - \frac{1}{2}k\,2\tilde{r} =$

$= mr\dot{\theta}^2 + mg\cos\theta - k\tilde{r}$

$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m\,2\dot{r} = m\dot{r}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

(1) $\boxed{m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k\tilde{r} = 0}$ $\overset{l_0}{\circledast}$ $\boxed{m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r - l_0) = 0}$

↳ $r(t)$, $\theta(t)$ ⟶ coordenadas acopladas

(θ) $\frac{\partial L}{\partial \theta} = -mgr\sin\theta$ $\quad \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mr^2\,2\dot{\theta} = mr^2\dot{\theta}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta = 0$$

(2) $\boxed{2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + gr\sin\theta = 0}$ ⟶ $\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} + g\sin\theta = 0}$

⊛ Casos simples conocidos :

(I) $\theta = cte = \theta_0 \implies$ oscilador armónico

$\theta = cte \implies \dot{\theta} = \ddot{\theta} = 0$

De (1) $\longrightarrow$ $\boxed{m\ddot{r} = -kr + mg\cos\theta}$

Oscilador armónico desplazado del equilibrio (oscila en torno a una posición de equilibrio distinta)

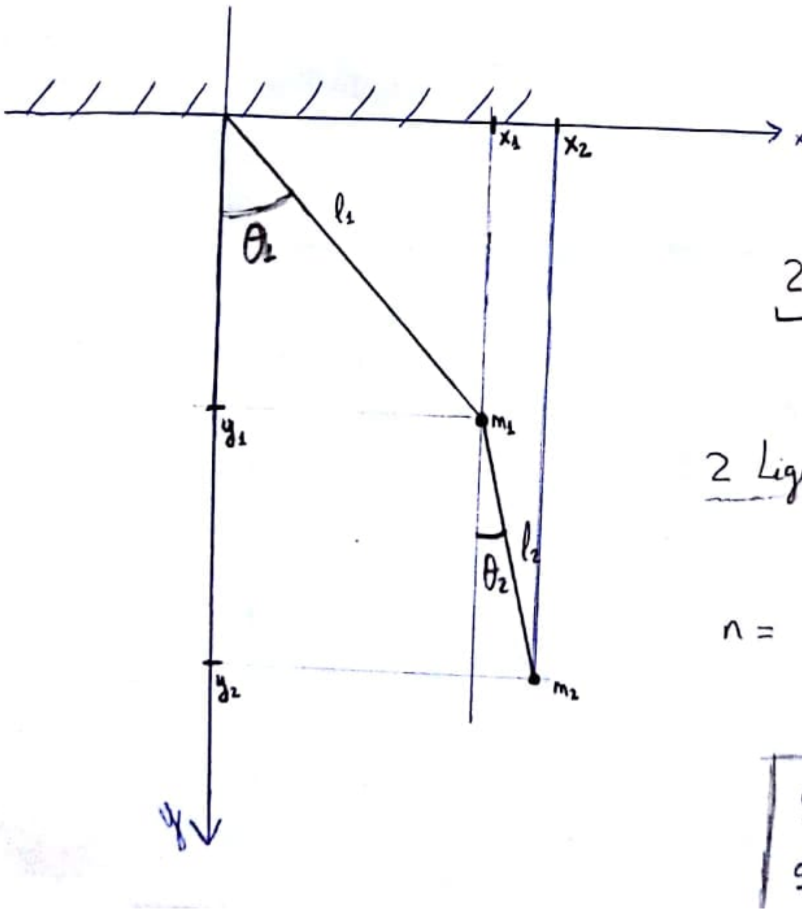
(II) $r = cte = r_0 \implies$ péndulo simple

$r = cte \implies \dot{r} = \ddot{r} = 0$

De (2) $\longrightarrow$ $r_0^2\,\ddot{\theta} + g\,r_0 \sin\theta = 0$

$\boxed{\ddot{\theta} = -\frac{g}{r_0}\sin\theta} \longrightarrow$ péndulo simple con $\ell = r_0$

b) Péndulo doble :



$\boxed{V=0}$

2 partículas $\longrightarrow$ 2D

$N = 2\cdot 2 = 4$ $\begin{cases}(x_1, y_1)\\(x_2, y_2)\end{cases}$

2 Ligaduras $\begin{cases} x_1^2 + y_1^2 = \ell_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell_2^2 \end{cases}$

$n = 4 - 2 = \boxed{2\ g.\ell.}$

2 coordenadas generalizadas :

$$\boxed{\begin{aligned} q_1 &= \theta_1 = \arctan\frac{x_1}{y_1} \\ q_2 &= \theta_2 = \arctan\frac{x_2 - x_1}{y_2 - y_1}\end{aligned}}$$

2 Ejs

$$x_1 = l_1 \cdot \sin\theta_1 \longrightarrow \dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1 \rightarrow \dot{x}_1^2 = l_1^2 \dot{\theta}_1^2 \cos^2\theta_1$$

$$y_1 = -l_1 \cdot \cos\theta_1 \longrightarrow \dot{y}_1 = l_1 \sin\theta_1 \dot{\theta}_1 \rightarrow \dot{y}_1^2 = l_1^2 \dot{\theta}_1^2 \sin^2\theta_1$$

$$x_2 = x_1 + l_2 \cdot \sin\theta_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = y_1 - l_2 \cos\theta_2 = -(l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2$$

$$\dot{x}_2^2 = l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_2^2 \dot{\theta}_2^2 \cos^2\theta_2 + 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\dot{y}_2 = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2$$

$$\dot{y}_2^2 = l_1^2 \dot{\theta}_1 \sin^2\theta_1 + l_2^2 \dot{\theta}_2^2 \sin^2\theta_2 + 2 l_1 l_2 \sin\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2$$

$$T = \frac{1}{2}\left[m_1 (\dot{x}_1^2 + \dot{y}_1^2) + m_2 (\dot{x}_2^2 + \dot{y}_2^2)\right] =$$

$$= \frac{1}{2}\left[m_1\left[l_1^2 \dot{\theta}_1^2\right] + m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]\right]$$

$$V = m_1 g y_1 + m_2 g y_2 = -\left(m_1 l_1 \cos\theta_1 + m_2 \left[l_1 \cos\theta_1 + l_2 \cos\theta_2\right]\right) g$$

$$\boxed{L = T - V}$$

$$L = \frac{1}{2}\left[(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + m_2 \left(l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)\right] +$$

$$+ g\left(m_1 l_1 \cos\theta_1 + m_2 \left[l_1 \cos\theta_1 + l_2 \cos\theta_2\right]\right)$$

Ecs. Lagrange: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right) - \dfrac{\partial L}{\partial q_j} = 0$

$\boxed{\theta_1}$

$$\frac{\partial L}{\partial \theta_1} = \frac{1}{2} m_2 \, 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left(-\sin(\theta_1 - \theta_2)\right) - \left(g m_1 l_1 \sin\theta_1 + g m_2 l_1 \sin\theta_1\right) =$$

$$= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g l_1 \sin\theta_1 (m_1 + m_2)$$

$$\frac{\partial L}{\partial \dot\theta_1} = \frac{1}{\cancel{2}}(m_1+m_2)\, l_1^2\, \cancel{2}\, \dot\theta_1 + \frac{1}{\cancel{2}}\cancel{2}\, m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1-\theta_2) =$$

$$= (m_1+m_2)\, l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = (m_1+m_2)\, l_1^2\, \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) +$$

$$+ m_2 l_1 l_2 \dot\theta_2 \left(-\sin(\theta_1-\theta_2)\cdot[\dot\theta_1-\dot\theta_2]\right) =$$

$$= (m_1+m_2)\, l_1^2\, \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) -$$

$$- m_2 l_1 l_2 \dot\theta_2 \dot\theta_1 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2)$$

$$\boxed{(m_1+m_2)\, l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + g l_1 \sin\theta_1 (m_1+m_2) = 0}$$

$\boxed{\theta_2}$

$$\frac{\partial L}{\partial \theta_2} = \frac{1}{\cancel{2}} m_2\, \cancel{2}\, l_1 l_2 \dot\theta_1 \dot\theta_2 \left(-\sin(\theta_1-\theta_2)\,(-1)\right) =$$

$$= m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - g m_2 l_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \dot\theta_2} = \frac{1}{\cancel{2}} m_2 l_2^2\, \cancel{2}\, \dot\theta_2 + \frac{1}{2} m_2\, 2\, l_1 l_2 \dot\theta_1 \cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1-\theta_2)\,[\dot\theta_1-\dot\theta_2]$$

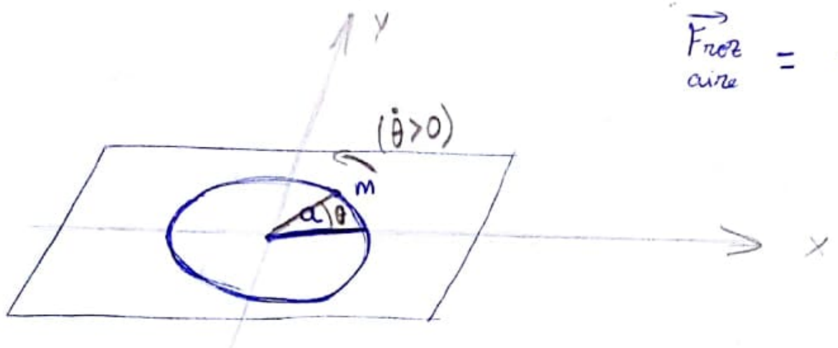
$$= m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2)$$

$$\boxed{m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) + g m_2 l_2 \sin\theta_2 = 0}$$

2 Ejs

(2.4) partícula m $\longrightarrow$ se mueve en una circunferencia horizontal de radio a.

$\vec{F}_{\substack{roz\\aire}} = -\alpha\, v^2\, \hat{v}$



1 partícula 2D $\longrightarrow$ N = 2
1 ligadura $x^2 + y^2 = a^2$

$\Big\}$ n = 1 g.l. $\longrightarrow$ coordenada generalizada $q_1 = \theta$

$$\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases} \quad \begin{cases} \dot{x} = -a\sin\theta\,\dot{\theta} \\ \dot{y} = a\cos\theta\,\dot{\theta} \end{cases}$$

$$\hat{v} = \frac{d\vec{r}/dt}{v} = \frac{(-a\sin\theta\dot{\theta},\, a\cos\theta\dot{\theta})}{a\dot{\theta}} = (-\sin\theta,\, \cos\theta)$$

$$\boxed{v^2 = \dot{x}^2 + \dot{y}^2} \qquad \begin{cases} \dot{x}^2 = a^2\sin^2\theta\,\dot{\theta}^2 \\ \dot{y}^2 = a^2\cos^2\theta\,\dot{\theta}^2 \end{cases}$$

$$v^2 = a^2\dot{\theta}^2$$

$$\boxed{v = a\dot{\theta}}$$

$$\vec{F}_r = -\alpha\, a^2\dot{\theta}^2(-\sin\theta,\, \cos\theta)$$

$$Q_\theta = \vec{F}_r \cdot \frac{\partial\vec{r}}{\partial\theta} = -\alpha a^2\dot{\theta}^2(-\sin\theta, \cos\theta)\cdot(-a\sin\theta,\, a\cos\theta) =$$

$$= -\alpha a^3\dot{\theta}^2(\sin^2\theta + \cos^2\theta) = -\alpha a^3\dot{\theta}^2$$

$$\boxed{Q_\theta = -\alpha a^3\dot{\theta}^2} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = Q_\theta$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\, a^2\dot{\theta}^2$$

$$\frac{\partial T}{\partial\theta} = 0, \quad \frac{\partial T}{\partial\dot{\theta}} = ma^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) = ma^2\ddot{\theta}$$

$$ma^2\ddot{\theta} = -\alpha a^3\dot{\theta}^2, \qquad \boxed{\ddot{\theta} = -\frac{\alpha a}{m}\dot{\theta}^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\alpha}{m} a \left(\frac{d\theta}{dt}\right)^2 \rightsquigarrow \text{red. de orden} \quad \frac{d\theta}{dt} = \varphi$$

$$\frac{d\varphi}{dt} = -\frac{\alpha a}{m}\varphi^2 \quad , \quad -\int \frac{d\varphi}{\varphi^2} = \frac{\alpha a}{m}\int dt + C_1$$

$$\frac{1}{\varphi} = \frac{\alpha a}{m} t + C_1$$

$$\varphi = \frac{1}{\frac{\alpha a}{m} t + C_1}$$

$$\frac{d\theta}{dt} = \frac{1}{\frac{\alpha a}{m} t + C_1} \quad , \quad \int_{\theta_0}^{\theta} d\theta' = \int_0^t \frac{dt'}{\frac{\alpha a}{m} t + C_1}$$

$$\theta - \theta_0 = \left(\ln\left(\frac{\alpha a}{m} t + C_1\right)\right]_0^t \frac{m}{\alpha a}$$

$$\theta - \theta_0 = \frac{m}{\alpha a}\left[\ln\left(\frac{\alpha a}{m} t + C_1\right) - \ln C_1\right]$$

$$\theta - \theta_0 = \frac{m}{\alpha a} \ln\left(\frac{\alpha a}{m C_1} t + 1\right)$$

$$\circledast\ \varphi(t) = \dot{\theta}(t) = \frac{1}{\frac{\alpha a}{m^2} t + C_1}$$

$$V_0 = \dot{\theta}(0) \cdot a = \frac{a}{C_1}$$

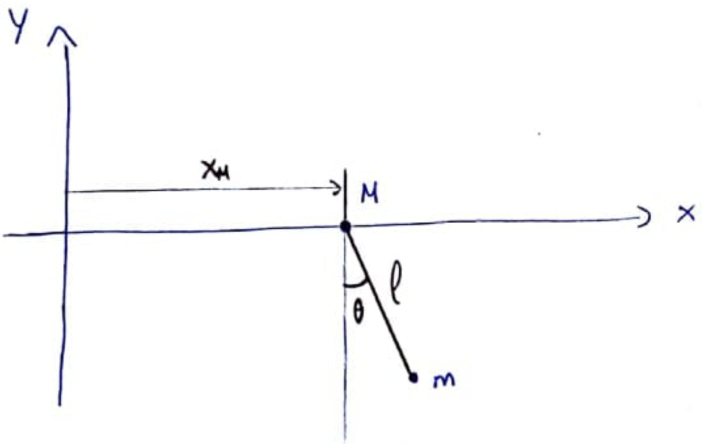
$$\boxed{\theta(t) = \theta_0 + \frac{m}{\alpha a} \ln\left(\frac{\alpha a}{m C_1} t + 1\right)}$$

$$\boxed{\theta(t) = \theta_0 + \frac{m}{\alpha a} \ln\left(\frac{\alpha V_0}{m} t + 1\right)}$$

Scanned by CamScanner

2.5 péndulo masa m, longitud ℓ

colgado de M, que se mueve sin rozamiento sobre una recta horizontal.



$\boxed{V=0}$

2 partículas 2D $\longrightarrow$ $N = 4$ $\begin{cases}(x_M, y_M)\\(x_m, y_m)\end{cases}$

2 Ligaduras : $\begin{array}{l} y_M = 0 \\ (x_m - x_M)^2 + y_m^2 = \ell^2 \end{array}$

$\boxed{n = 2 \text{ g.l.}}$

$\downarrow$

2 coordenadas generalizadas

$$\boxed{\begin{array}{l} q_1 = x_M = x \\ q_2 = \theta \end{array}}$$

$$\begin{cases} x_M = x \\ y_M = 0 \\ x_m = x + \ell \sin\theta \\ y_m = -\ell\cos\theta \end{cases} \qquad \begin{cases} \dot{x}_M = \dot{x} \\ \dot{y}_M = 0 \\ \dot{x}_m = \dot{x} + \ell\dot{\theta}\cos\theta \\ \dot{y}_m = \ell\dot{\theta}\sin\theta \end{cases}$$

$$T = \frac{1}{2} M (\dot{x}_M^2 + \dot{y}_M^2) + \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2) =$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[\dot{x}^2 + \ell^2\dot{\theta}^2\cos^2\theta + 2\dot{x}\ell\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\sin^2\theta \right] =$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \ell^2\dot{\theta}^2 + m\dot{x}\ell\dot{\theta}\cos\theta$$

$$V = Mg\cancelto{0}{y_M} + m g y_m = -mg\ell\cos\theta \qquad \boxed{V(y=0) = 0}$$

$$\boxed{L = T - V}$$

$$L = \frac{1}{2}(m+M)\dot{x}^2 + m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

Ecuaciones de Lagrange: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$

(x) $\frac{\partial L}{\partial x} = 0$ (x cíclica) $\Longrightarrow$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(p_x) = 0 \Rightarrow p_x = cte$

(Integral del movimiento)

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m+M)\dot{x} + m\ell\dot{\theta}\cos\theta = cte$$

d/dt (ec. Lagrange)

$$\boxed{(m+M)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = 0} \quad (1)$$

(θ) $\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta - m\ell\dot{x}\dot{\theta}\sin\theta = -m\ell\sin\theta(g + \dot{x}\dot{\theta})$

$\frac{\partial L}{\partial \dot{\theta}} = m\ell\dot{x}\cos\theta + m\ell^2\dot{\theta}$ $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell\ddot{x}\cos\theta - m\ell\dot{x}\sin\theta\dot{\theta} + m\ell^2\ddot{\theta}$

$$m\ell\ddot{x}\cos\theta + m\ell^2\ddot{\theta} + m\ell\sin\theta\, g = 0$$

θ no cíclica

⇓

$p_\theta \neq cte$

$$\boxed{\cos\theta\,\ddot{x} + \ell\ddot{\theta} + g\sin\theta = 0} \quad (2)$$

(1) $\ddot{x} = \frac{m\ell(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)}{m+M}$

(2)

$$\frac{\cos\theta\, m\ell\,(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)}{m+M} + \ell\ddot{\theta} + g\sin\theta = 0$$

$$\boxed{\ell\ddot{\theta}\left[1-\left(\frac{m}{m+M}\right)\cos^2\theta\right]+\left(\frac{m}{m+M}\right)\ell\dot{\theta}^2\sin\theta\cos\theta+g\sin\theta=0} \quad (3)$$

Ecuación desacoplada (en $\theta(t)$)

⊛ Aproximación para oscilaciones pequeñas : $\theta \to 0$

$$\boxed{\begin{aligned}\cos\theta &\approx 1\\ \sin\theta &\approx \theta\end{aligned}}$$

(3) :

$$\boxed{\ell\ddot{\theta}\left[1-\frac{m}{m+M}\right]+\frac{m}{m+M}\ell\dot{\theta}^2\theta+g\theta=0}$$

⊛ Cuando $x = cte = x_0$ $\Rightarrow$ Péndulo simple : $(x = cte \Rightarrow \ddot{x}=0)$

$$(2) \longrightarrow \ell\ddot{\theta}+g\sin\theta=0$$

$$\boxed{\ddot{\theta}=-\frac{g}{\ell}\sin\theta}$$

⊛ Cuando $m \ll M$ $\Longrightarrow$ $\frac{m}{m+M}\approx\frac{m}{M}\approx 0$

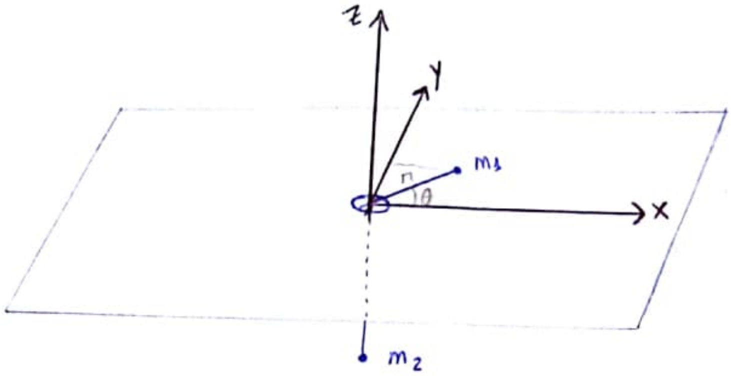
$$(3) \to \ell\ddot{\theta}+g\sin\theta=0$$

y para $\theta \simeq 0$ (oscilaciones pequeñas).

$$\boxed{\ell\ddot{\theta}+g\theta=0} \longrightarrow \text{Oscilador armónico}$$

$$\ddot{\theta}=-\frac{g}{\ell}\theta$$

2.6



2 partículas 3D $\longrightarrow$ $N = 2 \cdot 3 = 6 \begin{cases} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \end{cases}$

4 ligaduras : $z_1 = 0 \qquad y_2 = 0$

$x_2 = 0 \qquad \sqrt{x_1^2 + y_1^2} + z_2 = \ell \longrightarrow$ hilo inextensible

$n = 6 - 4 = 2$ g.l. $\longrightarrow$ 2 coordenadas generalizadas : $\boxed{\begin{matrix} q_1 = r \\ q_2 = \theta \end{matrix}}$

$$\begin{cases} x_1 = r\cos\theta \\ y_1 = r\sin\theta \\ z_2 = -(\ell - r) = r - \ell \end{cases} \qquad \begin{cases} \dot{x}_1 = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y}_1 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ \dot{z}_2 = \dot{r} \end{cases}$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{z}_2^2) =$$

$$= \frac{1}{2} m_1 (\dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta)$$

$$+ \frac{1}{2} m_2 \dot{r}^2 = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_2 \dot{r}^2 = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\theta}^2$$

$$V = 0 + m_2 g z_2 = m_2 g (r - \ell)$$

$\boxed{V(z=0) = 0}$

$\boxed{L = T - V}$ $\qquad \boxed{L = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\theta}^2 + m_2 g (\ell - r)}$

Ecs. de Lagrange: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$

(r) $\quad \frac{\partial L}{\partial r} = m_1 r \dot{\theta}^2 - m_2 g \qquad \frac{\partial L}{\partial \dot{r}} = (m_1 + m_2)\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = (m_1 + m_2)\ddot{r}$

$\boxed{(m_1 + m_2)\ddot{r} = m_1 r \dot{\theta}^2 - m_2 g}$ (1) $\longrightarrow$ peso m_2

F. centrípeta $[|m\vec{\omega} \times (\vec{\omega} \times \vec{r})|]$

(θ) $\quad \frac{\partial L}{\partial \theta} = 0 \Rightarrow \theta$ cíclica $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0$

$$\frac{\partial L}{\partial \dot\theta} = P_\theta = cte$$

$$P_\theta = \frac{\partial L}{\partial \dot\theta} = m_1 r^2 \dot\theta = cte \rightsquigarrow \boxed{\underset{(cte)}{P_\theta} = m_1 r^2 \dot\theta} \quad (2)$$

Momento angular de m_1 en torno al origen

$$\vec{L}_1 = \vec{r}_1 \times \vec{p}_1 = r\hat{r} \times m_1(\dot r \hat r + r\dot\theta\hat\theta) = m_1 r^2\dot\theta(\hat r \times \hat\theta) = \underset{P_\theta}{\underset{\parallel}{m_1 r^2 \dot\theta}}\,\hat z$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \boxed{m_1\, 2r\dot r\dot\theta + m_1 r^2 \ddot\theta = 0}$$

$$\boxed{2\dot r\dot\theta + r\ddot\theta = 0}$$

Desacoplamos $\longrightarrow$ Sustituimos **(2)** en (1):

$$P_\theta = m_1 r^2\dot\theta \;,\; \boxed{\dot\theta = \frac{P_\theta}{m_1 r^2}} \longrightarrow \boxed{(m_1+m_2)\ddot r = \frac{P_\theta^2}{m_1 r^3} - m_2 g}$$

Ecuación del movimiento en (r)

$\longrightarrow$ Fuerzas conservativas $\Longrightarrow \boxed{E = cte}$

$\longrightarrow \frac{\partial L}{\partial t} = 0 \Longrightarrow \boxed{H = cte} \qquad H = \sum_{j=1}^{n} p_j \dot q_j - L$

(a) $\vec r(q_j)$ no dependen de t

(b) V no depende de $\dot r, \dot\theta$ $\quad \Big\} \Rightarrow \boxed{H = E}$

$$E = T + V = \frac{1}{2}(m_1+m_2)\dot r^2 + \frac{1}{2} m_1 r^2\dot\theta^2 + m_2 g(r-\ell)$$

$\boxed{\dot\theta = \frac{P_\theta}{m_1 r^2}}$

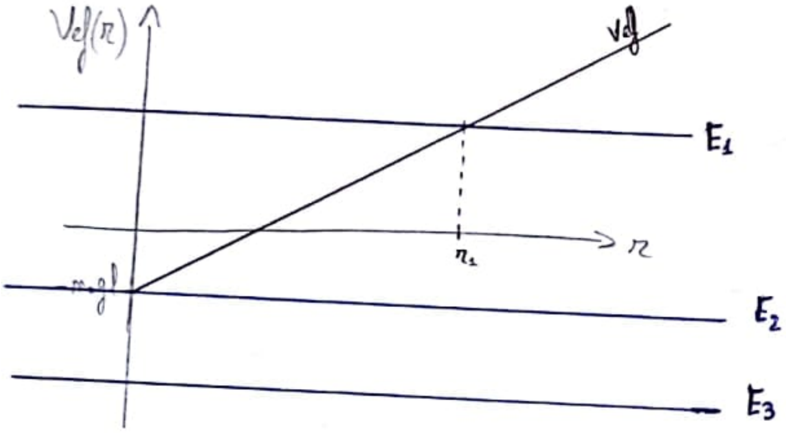
$$\boxed{E = \underbrace{\frac{1}{2}(m_1+m_2)\dot r^2}_{\text{Término cinético}} + \underbrace{\frac{1}{2}\frac{P_\theta^2}{m_1 r^2} + m_2 g(r-\ell)}_{\text{Vefectivo }(r)} = H = cte}$$

$$V_{ef}(r) = \frac{1}{2}\frac{P_\theta^2}{m_1 r^2} + m_2 g(r-\ell)$$

r permitidos $\rightarrow$ r / $E \geq V_{ef}(r)$ [porque el término cinético siempre es ≥ 0]

Ⓐ $\boxed{P_\theta = 0} \longrightarrow$ no hay momento angular, no hay rotación

$$V_{ef}(r) = m_2 g(r-\ell) = -m_2 g\ell + m_2 g r$$



① $E_1 > -m_2 g\ell$

$r \in [0, r_1]$

$\boxed{r = 0} \rightarrow$ equilibrio inestable

② $E_2 = -m_2 g\ell$

Solo puede estar en $\boxed{r = 0}$

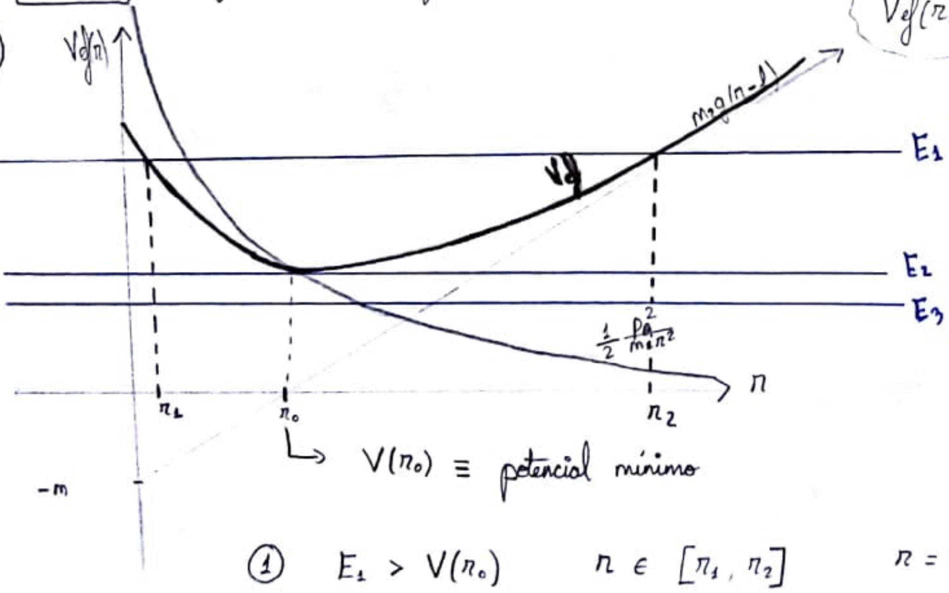
③ $E_3 < -m_2 g\ell$

No hay ninguna posición permitida

$\boxed{P_\theta \neq 0} \rightarrow$ hay momento angular, $\exists$ rotación

Ⓑ

$$V_{ef}(r) = \frac{P_\theta^2}{2m_1 r^2} + m_2 g(r-\ell)$$



$\hookrightarrow V(r_0) \equiv$ potencial mínimo

① $E_1 > V(r_0)$ $\quad r \in [r_1, r_2]$ $\quad r = r_0 \rightarrow$ eq. estable

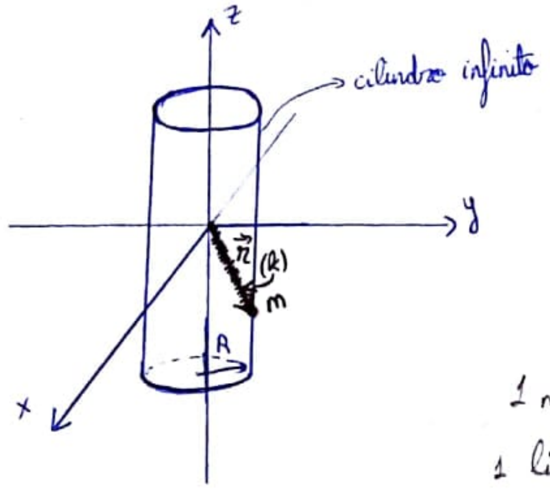
② $E_2 = V(r_0)$ $\quad r = r_0 \quad \forall t$

③ $E_3 < V(r_0)$ Ninguna posición permitida

2.7 Movimiento de una partícula que se mueve sobre la superficie de un cilindro vertical de radio R $\longrightarrow$ $\boxed{\vec{F} = -k\vec{r}}$

Fuerza elástica (conservativa)

$\boxed{k>0}$



Coordenadas cilíndricas
$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

1 masa en 3D $\longrightarrow N = 3 \rightarrow (\rho, \varphi, z)$

1 ligadura $\longrightarrow \boxed{\rho = R} \longrightarrow \boxed{n = 2 \text{ g.l.}}$

2 coordenadas generalizadas

$$\boxed{\begin{aligned} q_1 &= \varphi \\ q_2 &= z \end{aligned}}$$

$$\boxed{L = T - V}$$

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = z \end{cases} \qquad \begin{cases} \dot{x} = -R\sin\varphi\,\dot{\varphi} \\ \dot{y} = R\cos\varphi\,\dot{\varphi} \\ \dot{z} = \dot{z} \end{cases}$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$T = \frac{1}{2}m\left(R^2\dot{\varphi}^2\sin^2\varphi + R^2\dot{\varphi}^2\cos^2\varphi + \dot{z}^2\right) = \frac{1}{2}m\left(R^2\dot{\varphi}^2 + \dot{z}^2\right)$$

$$V = \underbrace{mgz}_{} + \frac{1}{2}kr^2 = mgz + \frac{1}{2}k\left(z^2 + R^2\right)$$

$$\boxed{V_g(z=0) = 0}$$

$$L = \frac{1}{2}m\left(R^2\dot{\varphi}^2 + \dot{z}^2\right) - mgz - \frac{1}{2}k\left(z^2 + R^2\right)$$

Ecs Lagrange: $\boxed{\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right) - \dfrac{\partial L}{\partial q_j} = 0}$

(z) $\dfrac{\partial L}{\partial z} = -mg - kz \qquad \dfrac{\partial L}{\partial \dot{z}} = m\dot{z} = p_z$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z} \longrightarrow \boxed{m\ddot{z} = -mg - kz}$$

φ $\dfrac{\partial L}{\partial \varphi} = 0$, φ cíclica $\Longrightarrow$ $p_\varphi = \dfrac{\partial L}{\partial \dot{\varphi}} = cte$

$$\frac{\partial L}{\partial \dot{\varphi}} = mR^2\dot{\varphi} = p_\varphi = cte \Rightarrow \boxed{\dot{\varphi} = cte}$$

$p_\varphi \equiv$ componente z del momento angular $\vec{L}$.

$$\boxed{mR^2\ddot{\varphi} = 0}$$

$\longrightarrow$ $\dfrac{\partial L}{\partial t} = 0 \Longrightarrow$ H se conserva

$\longrightarrow$ (1) $\vec{r}(\varphi, z)$ no depende de t

(2) $V \neq V(\dot{z}, \dot{\varphi})$ $\Big\} \Rightarrow \boxed{H = E} \Longrightarrow$ E se conserva

$$H = \sum_{j=1}^{2} \frac{\partial L}{\partial \dot{q}_j} \cdot \dot{q}_j - L = \frac{\partial L}{\partial \dot{\varphi}} \dot{\varphi} + \frac{\partial L}{\partial \dot{z}} \dot{z} - L =$$

$$= p_\varphi \dot{\varphi} + p_z \dot{z} - L = \frac{1}{2}\left(mR^2\dot{\varphi}^2 + m\dot{z}^2\right) + mgz + \frac{1}{2}k(R^2 + z^2)$$

$$H = E = \frac{1}{2} m \left(R^2\dot{\varphi}^2 + \dot{z}^2\right) + mgz + \frac{1}{2}k\left(R^2 + z^2\right)$$

$$(*)\ \dot{\varphi} = \frac{p_\varphi}{mR^2} \longrightarrow H = E = \frac{1}{2} m \left(\frac{p_\varphi^2}{m^2R^2} + \dot{z}^2\right) + mgz + \frac{1}{2}k\left(R^2 + z^2\right)$$

$$E = \underbrace{\frac{1}{2} m\dot{z}^2}_{\substack{\text{Término} \\ \text{cinético}}\ \geq 0} + \underbrace{\frac{p_\varphi^2}{2mR^2} + mgz + \frac{1}{2}k\left(R^2 + z^2\right)}_{V_{ef}(z)}$$

$$V_{ef} = \frac{p_\varphi^2}{2mR^2} + mgz + \frac{1}{2}k\left(R^2 + z^2\right)$$

$$\boxed{z_0 = -\frac{mg}{k}} < 0$$

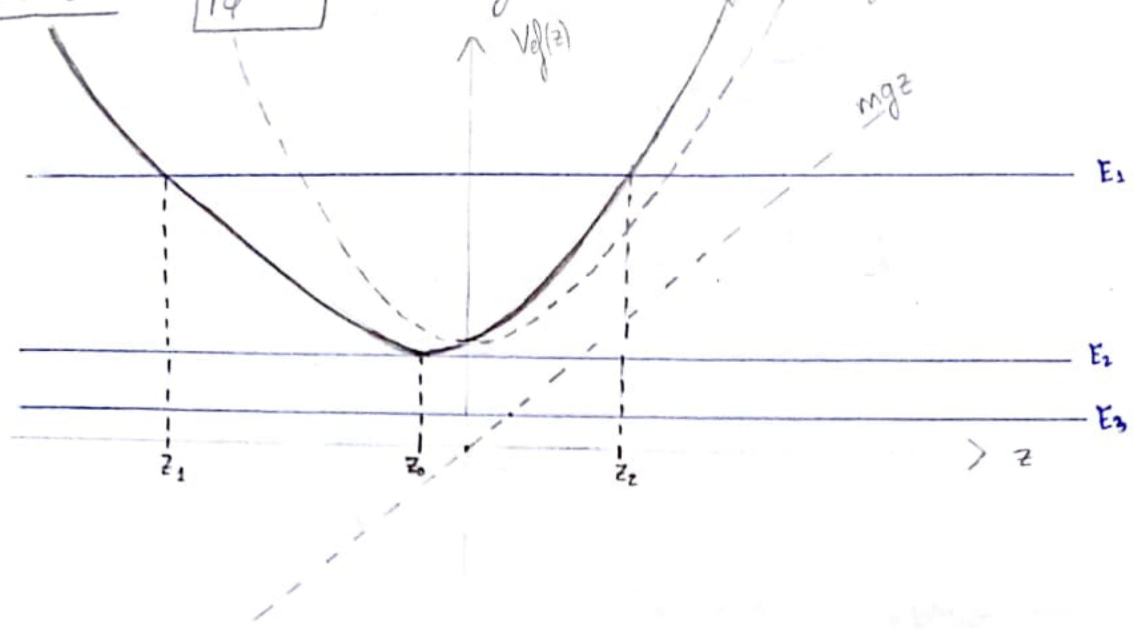
$$\frac{dV_{ef}}{dz} = mg + kz \longrightarrow \left.\frac{dV_{ef}}{dz}\right|_{z=z_0} = 0 = mg + kz_0$$

$V_{ef}(z_0) \equiv$ mínimo del potencial

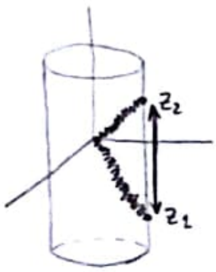
2 Ejs

z permitidos $\longrightarrow$ $z / \ E \geq V_{ef}(z)$

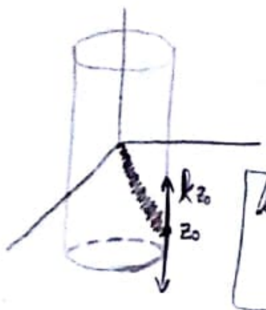
Caso I: $\boxed{P_\varphi = 0} \longrightarrow$ no gira $(\dot{\varphi} = 0)$, $V_{ef}(z) = mgz + \frac{1}{2}k(R^2 + z^2)$



Ⓐ $E = E_1 > V_{ef}(z_0) \Rightarrow z \in [z_1, z_2]$ $\quad z = z_0 \longrightarrow$ pto eq. estable



Ⓑ $E = E_2 = V_{ef}(z_0) \Rightarrow \underbrace{z = z_0}_{\text{parada, en equilibrio}} < 0$ $\left(\text{porque } z_0 = -\frac{mg}{k}\right)$

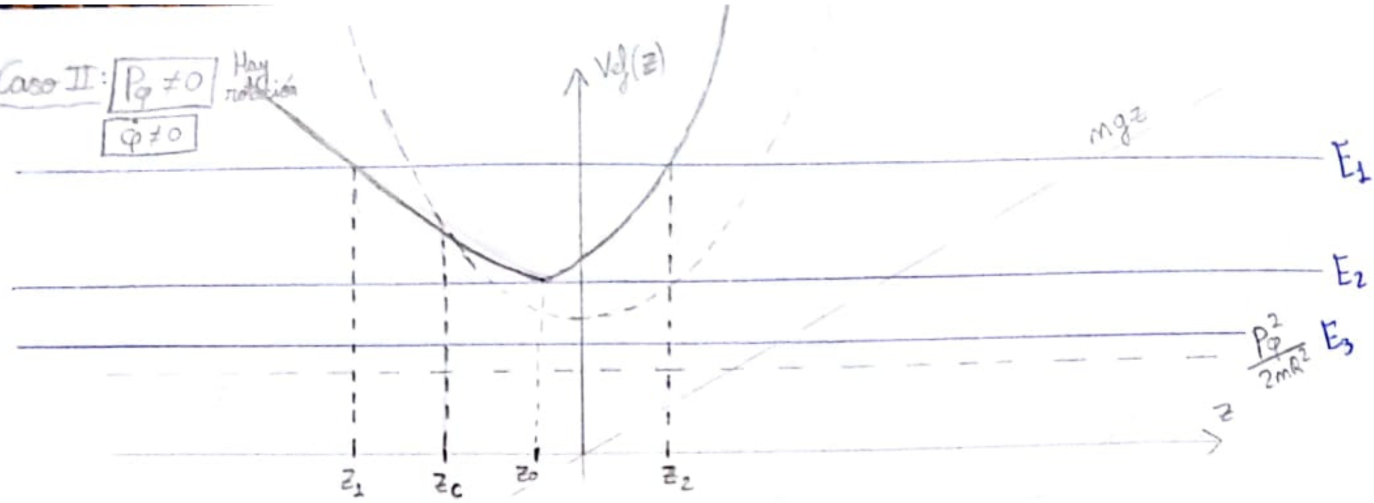


$\boxed{\begin{array}{c} kz_0 = mg \\ z_0 = -\frac{mg}{k} \end{array}} \longrightarrow \underset{eq.}{\sum F = 0} \rightarrow$ z_0 tiene que ser negativo, en la parte positiva ambas fuerzas van hacia abajo

Ⓒ $E = E_3 < V_{ef}(z_0) \longrightarrow$ No hay posiciones permitidas

Caso II: $\boxed{P_\varphi \neq 0}$ Hay rotación

$\boxed{\dot{\varphi} \neq 0}$

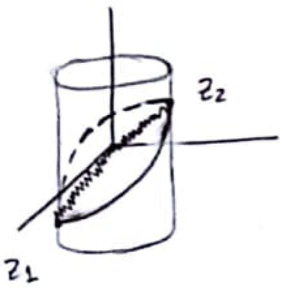


$$-mg z_c = \frac{P_\varphi^2}{2mR^2}$$

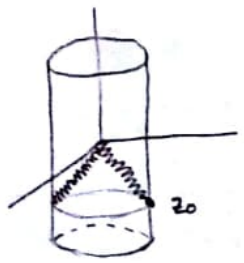
$$\boxed{z_c = \frac{-P_\varphi^2}{2m^2 g R^2}}$$

$$V_{ef}(z) = \frac{P_\varphi^2}{2mR^2} + mgz + \frac{1}{2}k(R^2 + z^2)$$

Ⓐ $E = E_1 > V_{ef}(z_0)$, $z \in [z_1, z_2] \longrightarrow z = z_0$ pto eq. estable



Ⓑ $E = E_2 = V_{ef}(z_0) \rightarrow z = z_0 < 0$ $\quad P_\varphi \neq 0,\ \dot{\varphi} \neq 0$



Ⓒ $E = E_3 < V_{ef}(z_0) \longrightarrow$ No hay posiciones permitidas

Ecuaciones del movimiento:

$\rightarrow\ p_\varphi = ma^2\dot{\varphi} = cte$

$$\left.\begin{array}{l}\dot{\varphi} = \omega_\varphi = cte \\ \ddot{\varphi} = 0\end{array}\right\} \rightarrow \boxed{\varphi(t) = \omega_\varphi t + \varphi_0} \qquad \varphi_0 = \varphi(t_0)$$

$\rightarrow\ m\ddot{z} = -kz - mg \rightarrow$ oscilador armónico en torno a z_0

$$\ddot{z} = -\frac{k}{m}z - g = -\frac{k}{m}\left(z + \frac{mg}{k}\right)$$

$$\boxed{(*)\ \text{c.v. } \eta = z + \frac{mg}{k},\ \dot{\eta} = \dot{z},\ \ddot{\eta} = \ddot{z}}$$

$$\ddot{\eta} = -\frac{k}{m}\eta \quad . \quad \boxed{\eta = A\sin(\omega_z t + \phi)} \qquad (*)\ \omega_z = \sqrt{\frac{k}{m}}$$

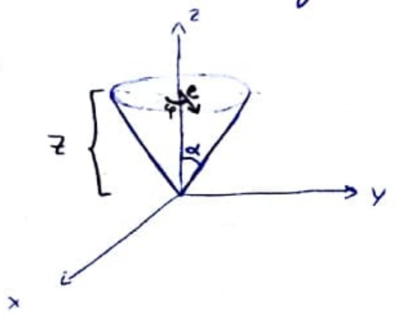
$$z + \frac{mg}{k} = A\sin(\omega_z t + \phi)$$

$$z = A\sin(\omega_z t + \phi) - \frac{mg}{k}$$

$(*)\ z_0 = -\frac{mg}{k}$

$$\boxed{z = A\sin(\omega_z t + \phi) + z_0} \rightarrow$$ Oscilador armónico en torno a z_0

(2.8) Movimiento de una partícula m sobre la superficie de un cono vertical invertido de semiángulo α sometida a la fuerza de la gravedad.



$$\begin{cases} e = z\tan\alpha \\ \varphi \\ z \end{cases}$$

$$\begin{cases} x = e\cos\varphi = z\tan\alpha\cos\varphi \\ y = e\sin\varphi = z\tan\alpha\sin\varphi \\ z = z \end{cases}$$

$$\begin{cases} \dot{x} = \dot{z}\tan\alpha\cos\varphi - \dot{\varphi}\sin\varphi\, z\tan\alpha \\ \dot{y} = \dot{z}\tan\alpha\sin\varphi + \dot{\varphi}\cos\varphi\, z\tan\alpha \\ \dot{z} = \dot{z} \end{cases}$$

$\left.\begin{array}{l}\text{3 coord.} \\ \text{1 ligadura}\end{array}\right\} n = 2$ g.l.

$\hookrightarrow$ 2 coordenadas generalizadas $\begin{cases} q_1 = z \\ q_2 = \varphi \end{cases}$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$T = \frac{1}{2} m \left[\tan^2\alpha \left[(\dot{z}\cos\varphi - \dot{\varphi} z \sin\varphi)^2 + (\dot{z}\sin\varphi + \dot{\varphi} z \cos\varphi)^2 \right] + \dot{z}^2 \right] =$$

$$= \frac{1}{2} m \left[\tan^2\alpha \left(\dot{z}^2\cos^2\varphi + \dot{\varphi}^2 z^2 \sin^2\varphi - 2z\dot{z}\dot{\varphi}\cos\varphi\sin\varphi + \dot{z}^2\sin^2\varphi + \dot{\varphi}^2 z^2 \cos^2\varphi + \right.\right.$$

$$\left.\left. + 2z\dot{z}\dot{\varphi}\sin\varphi\cos\varphi \right) + \dot{z}^2 \right] = \frac{1}{2} m \left[\tan^2\alpha \left(\dot{z}^2 + \dot{\varphi}^2 z^2 \right) + \dot{z}^2 \right]$$

$$V = mgz$$

$$\boxed{V(z=0)=0} \qquad \boxed{L = T - V}$$

$$\boxed{L = \frac{1}{2} m \left[\dot{z}^2 \underbrace{(1+\tan^2\alpha)}_{1/\cos^2\alpha} + \dot{\varphi}^2 z^2 \tan^2\alpha \right] - mgz}$$

Ecs Lagrange: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

φ

$$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi \text{ cíclica} \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = p_\varphi = cte$$

$$\frac{\partial L}{\partial \dot{\varphi}} = m\dot{\varphi} z^2 \tan^2\alpha = p_\varphi = cte \Rightarrow \dot{\varphi} = cte$$

$$p_\varphi = m\dot{\varphi} z^2 \tan^2\alpha \,, \quad \dot{\varphi} = \frac{p_\varphi}{m z^2 \tan^2\alpha}$$

z

$$\frac{\partial L}{\partial z} = m\dot{\varphi}^2 z \tan^2\alpha - mg$$

$$\frac{\partial L}{\partial \dot{z}} = \frac{m\dot{z}}{\cos^2\alpha} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{m\ddot{z}}{\cos^2\alpha}$$

$$\frac{m}{\cos^2\alpha}\ddot{z} - m\dot{\varphi}^2 z \tan^2\alpha + mg = 0$$

$$\boxed{\frac{1}{\cos^2\alpha}\ddot{z} - \dot{\varphi}^2 \tan^2\alpha\, z + g = 0}$$

$$\ddot{z} - \left(\frac{p_\varphi}{m z^2 \tan^2\alpha}\right)^2 \tan^2\alpha \cos^2\alpha + g\cos^2\alpha = 0$$

$$\ddot{z} - \frac{p_\varphi^2}{m^2 z^3 \tan^2\alpha}\cos^2\alpha + g\cos^2\alpha =$$

Desacop

$$\boxed{\ddot{z} + \left(g - \frac{p_\varphi^2}{m^2 z^3 \tan^2\alpha}\right)\cos^2\alpha = 0}$$

$\rightarrow \dfrac{\partial L}{\partial t} = 0 \Rightarrow H = cte$

$\rightarrow$ Fuerzas conservativas $\Rightarrow E = cte$

$$\left.\begin{array}{l} (1)\ \vec{r}(z,\varphi),\ \text{no depende explícitamente de } t. \\ (2)\ V \neq V(\dot{z},\dot{\varphi}) \end{array}\right\} \Rightarrow \boxed{H = E}$$

$$E = T + V \longrightarrow E = \frac{1}{2} m \left[\frac{1}{\cos^2\alpha} \dot{z}^2 + \dot{\varphi}^2 z^2 \tan^2\alpha \right] + mgz$$

$$(*)\ \dot{\varphi} = \frac{P_\varphi}{m z^2 \tan^2\alpha} \longrightarrow E = \frac{1}{2} m \left[\frac{\dot{z}^2}{\cos^2\alpha} + \frac{P_\varphi^2 z^2 \tan^2\alpha}{m^2 z^4 \tan^4\alpha} \right] + mgz$$

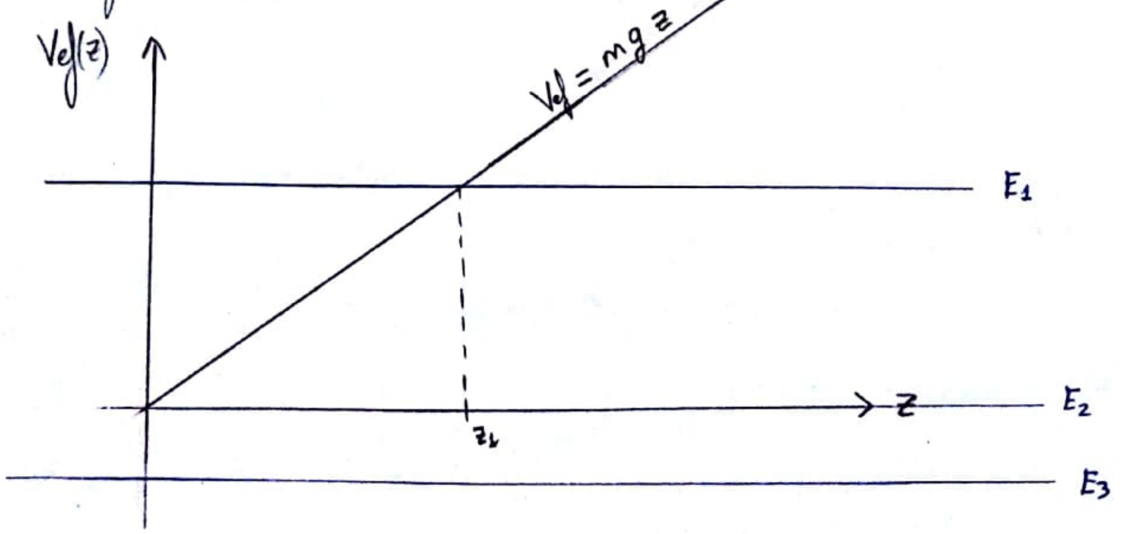
$$E = \frac{1}{2} m \left[\frac{\dot{z}^2}{\cos^2\alpha} + \frac{P_\varphi^2}{m^2 z^2 \tan^2\alpha} \right] + mgz$$

$$\boxed{E = \underbrace{\frac{1}{2} m \frac{\dot{z}^2}{\cos^2\alpha}}_{\text{Término cinético } (\geq 0)} + \underbrace{\frac{P_\varphi^2}{2 m z^2 \tan^2\alpha} + mgz}_{V_{ef}(z)}}$$

$$V_{ef}(z) = \overbrace{\frac{P_\varphi^2}{2m\tan^2\alpha}}^{cte \geq 0} \frac{1}{z^2} + \overbrace{mg}^{cte > 0}\, z \longrightarrow \text{Permitidos } z \,/\, V_{ef}(z) \leq E$$

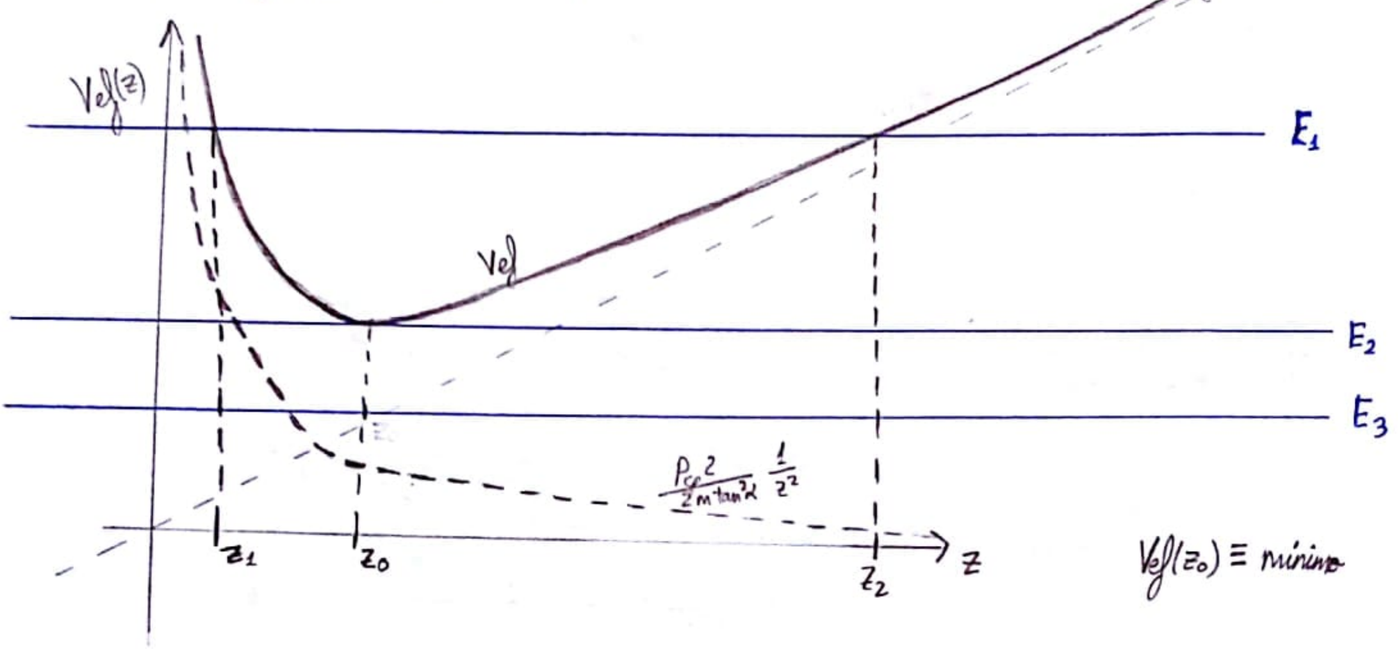
Caso I: $\boxed{P_\varphi = 0} \longrightarrow$ No hay rotación $\dot{\varphi} = 0$

$V_{ef}(z) = mg\cos^2\alpha\ z$ $\quad (z \geq 0)$



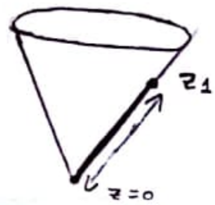
Caso II: $P_\varphi \neq 0 \Rightarrow \dot{\varphi} \neq 0$ Hay rotación

$$V_{ef}(z) = \frac{P_\varphi^2}{2m\tan^2\alpha}\,\frac{1}{z^2} + mgz$$



$V_{ef}(z_0) \equiv$ mínimo

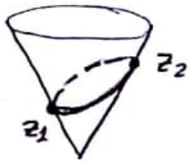
I (A) $E = E_1 > 0 \Rightarrow z \in [0, z_1]$



(B) $E = E_2 = 0 \Rightarrow z = 0$ quieto

(C) $E = E_3 < 0 \Rightarrow$ Ninguna posición permitida

II (A) $E = E_1 > V_{ef}(z_0)$, $z \in [z_1, z_2]$



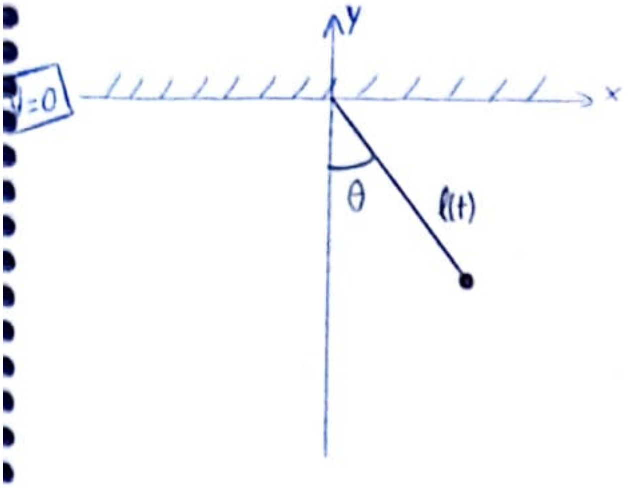
(B) $E = E_2 = V_{ef}(z_0)$, $z = z_0$ eq. estable

(C) $E = E_3 < V_{ef}(z_0)$, Ninguna posición permitida

2 Ejs

(29) Péndulo simple plano $\longrightarrow$ m, ℓ

se acorta a velocidad cte desde que el péndulo empieza a moverse



$$\boxed{\frac{d\ell}{dt} = -\alpha = cte} \quad (\alpha > 0)$$

$$\int_{\ell_0}^{\ell} d\ell = -\alpha \int_0^t dt$$

$$\boxed{\ell = \ell_0 - \alpha t}$$

1 partícula en 2D $\longrightarrow (x, y)$
1 ligadura $x^2 + y^2 = \ell^2(t)$ (reónoma)

$$\boxed{g_L = 0}$$

$$\begin{cases} x = (\ell_0 - \alpha t)\sin\theta \\ y = -(\ell_0 - \alpha t)\cos\theta \end{cases}$$

$$\dot{x} = -\alpha \sin\theta + (\ell_0 - \alpha t)\cos\theta\,\dot{\theta}$$
$$\dot{y} = \alpha \cos\theta + (\ell_0 - \alpha t)\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) =$$

$$= \frac{1}{2} m \left[\alpha^2 \sin^2\theta + (\ell_0 - \alpha t)^2 \cos^2\theta\,\dot{\theta}^2 - 2\alpha(\ell_0 - \alpha t)\sin\theta\cos\theta\,\dot{\theta} + \right.$$
$$\left. + \alpha^2\cos^2\theta + (\ell_0 - \alpha t)^2 \sin^2\theta\,\dot{\theta}^2 + 2\alpha(\ell_0 - \alpha t)\dot{\theta}\sin\theta\cos\theta\right] =$$

$$= \frac{1}{2} m \left[\alpha^2 + (\ell_0 - \alpha t)^2\dot{\theta}^2\right]$$

$$V = mgy = -mg(\ell_0 - \alpha t)\cos\theta$$
$(V_{y=0}) = 0)$

$$L = T - V$$

$$\boxed{L = \frac{1}{2} m \left[\alpha^2 + (\ell_0 - \alpha t)^2\dot{\theta}^2\right] + mg(\ell_0 - \alpha t)\cos\theta}$$

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L = p_\theta \dot{q}_\theta - L = \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} - L =$$

$$= m(l_0 - \alpha t)^2 \dot{\theta}^2 - \frac{1}{2} m \left[\alpha^2 + (l_0 - \alpha t)^2 \dot{\theta}^2\right] - mg(l_0 - \alpha t)\cos\theta$$

$$= \frac{1}{2} m (l_0 - \alpha t)^2 \dot{\theta}^2 - \frac{1}{2} m \alpha^2 - mg(l_0 - \alpha t)\cos\theta$$

$H \neq cte \Rightarrow$ $\boxed{\text{H no se conserva}}$ $\left(\frac{\partial L}{\partial t} \neq 0 \Rightarrow H \neq cte\right)$

$i \rightarrow (x, y) \longrightarrow (\theta)$ dependen de $t \longrightarrow \boxed{E \neq H}$

$$E = T + V = \frac{1}{2} m (l_0 - \alpha t)^2 \dot{\theta}^2 + \frac{1}{2} m\alpha^2 - mg(l_0 - \alpha t)\cos\theta$$

$E = E(t) \Longrightarrow$ $\boxed{\text{E no se conserva}}$

2.10 Cuerpo puntual $\longrightarrow$ m

Resbala sin rozamiento por una $\boxed{\text{recta AB}}$ partiendo del origen θ.

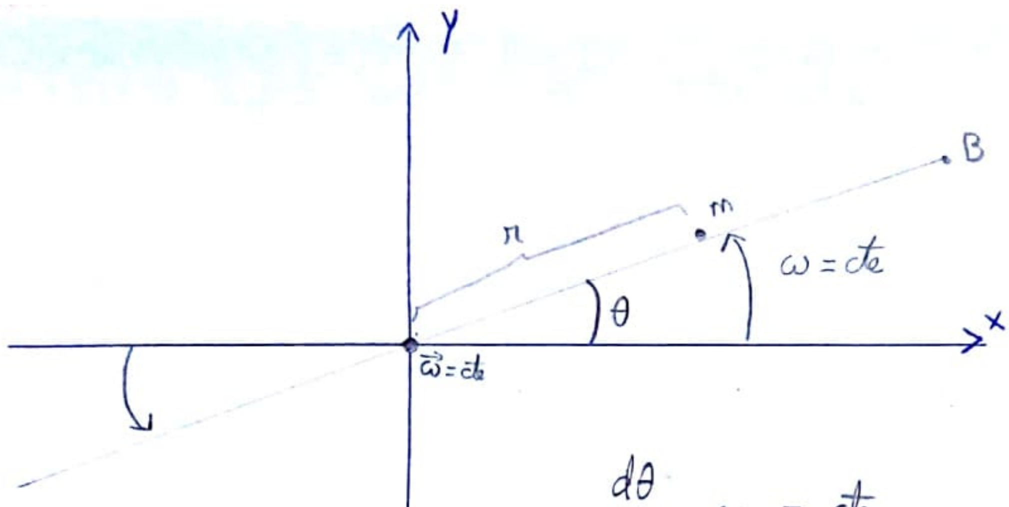
$\downarrow$

Se mueve en el plano XY con $\omega = cte$ alrededor de θ.

Gravedad en dirección $-Y$

(en $t = 0$ coincide con OX)

Ecuación del movimiento (integrada)



$$\frac{d\theta}{dt} = \omega = cte$$

$$\theta(t) = \omega t + \theta_0 \longrightarrow \boxed{\theta_0 = 0}$$

$$\boxed{\theta(t) = \omega t}$$

1 partícula 2D $\longrightarrow (x,y)$
1 ligadura $\theta(t) = \omega t$

$\boxed{n = 1 \, g.l.}$ $\boxed{q_1 = r}$

$$\begin{cases} x = r\cos\omega t \\ y = r\sin\omega t \end{cases} \longrightarrow \begin{cases} \dot{x} = \dot{r}\cos\omega t - r\sin\omega t\,\omega \\ \dot{y} = \dot{r}\sin\omega t + r\cos\omega t\,\omega \end{cases}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \left(\dot{r}^2\cos^2\omega t + r^2\omega^2\sin^2\omega t - \underline{2r\omega\dot{r}\cos\omega t\sin\omega t} + \right.$$

$$\left. + \dot{r}^2\sin^2\omega t + r^2\omega^2\cos^2\omega t + \underline{2r\omega\dot{r}\cos\omega t\sin\omega t} \right) =$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\omega^2)$$

$$V = mgy = mgr\sin\omega t$$

$V_{(y=0)} = 0$ $\boxed{L = T - V}$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\omega^2) - mgr\sin\omega t$$

Ec Lagrange: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0}$

$$\frac{\partial L}{\partial r} = m\omega^2 r - m g \sin\omega t$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\ddot{r} - \omega^2 r + g \sin\omega t = 0$$

$$\boxed{\ddot{r} - \omega^2 r = -g \sin\omega t}$$

1º la homogénea $\qquad \ddot{r} - \omega^2 r = 0$

$$\boxed{r = e^{kt}} \rightarrow k^2 - \omega^2 = 0$$

$$\boxed{k = \pm\omega}$$

$$r = C_1 e^{\omega t} + C_2 e^{-\omega t}$$

$$\boxed{r = A \sin\omega t} \longrightarrow \ddot{r} = -A\omega^2 \sin\omega t$$

$$-A\omega^2 \sin\omega t - A\omega^2 \sin\omega t = -g \sin\omega t$$

$$-2A\omega^2 = g$$

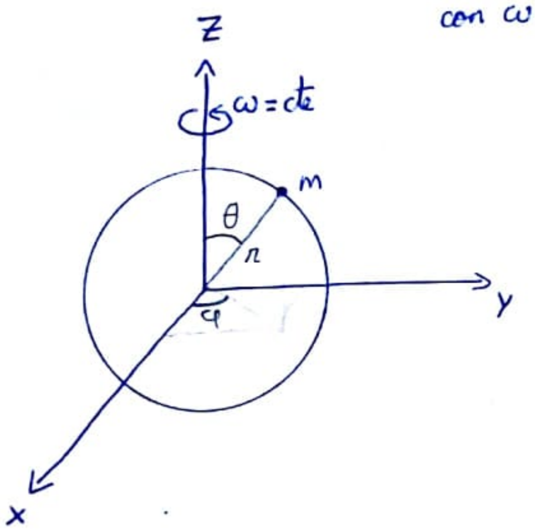
$$\boxed{A = \frac{-g}{2\omega^2}}$$

$$\boxed{r = C_1 e^{\omega t} + C_2 e^{-\omega t} - \frac{g}{2\omega^2} \sin\omega t}$$

2.11 Esfera masa $m \longrightarrow$ sin rozamiento

↳ aro circular de radio a.

en un plano vertical que gira alrededor de un diámetro vertical con $\omega =$ cte



1 partícula en 3D $\rightarrow N = 3$

2 ligaduras $\begin{cases} r = a \\ \varphi = \omega t + \varphi_0 \end{cases}$ $\Big\}$ 1 g.l. $\downarrow$ $\boxed{q_1 = \theta}$

$\boxed{\varphi_0 = 0}$ $\boxed{\dfrac{d\varphi}{dt} = \omega}$ $(\theta \in [0, \pi])$

$$\begin{cases} x = a \cdot \cos\omega t \cdot \sin\theta \\ y = a \sin\omega t \cdot \sin\theta \\ z = a \cos\theta \end{cases}$$

$$\begin{cases} \dot{x} = a \cdot \sin\theta\,(-\sin\omega t)\,\omega + a\cos\omega t \cos\theta\,\dot{\theta} \\ \dot{y} = a \sin\theta\,(\cos\omega t)\,\omega + a \sin\omega t \cos\theta\,\dot{\theta} \\ \dot{z} = -a\sin\theta\,\dot{\theta} \end{cases}$$

$$T = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m \Big[a^2 \sin^2\theta \sin^2\omega t\,\omega^2 + a^2 \cos^2\omega t \cos^2\theta\,\dot{\theta}^2 -$$
$$- \cancel{2a^2 \sin\theta\cos\theta\,\omega\dot{\theta}\,\sin\omega t\cos\omega t} +$$
$$+ a^2 \sin^2\theta \cos^2\omega t\,\omega^2 + a^2 \sin^2\omega t \cos^2\theta\,\dot{\theta}^2 +$$
$$+ \cancel{2a^2 \sin\theta\cos\theta\,\omega\dot{\theta}\,\sin\omega t\cos\omega t} +$$
$$+ a^2\dot{\theta}^2\sin^2\theta \Big] =$$

$$= \frac{1}{2} m \left[a^2 \sin^2\theta\,\omega^2 + a^2\cos^2\theta\,\dot{\theta}^2 + a^2\dot{\theta}^2\sin^2\theta \right] =$$
$$= \frac{1}{2} m \left[a^2\omega^2\sin^2\theta + a^2\dot{\theta}^2 \right]$$

$$V = mgz = mga\cos\theta$$

$$L = T - V = \frac{1}{2} m \left[a^2\omega^2 \sin^2\theta + a^2\dot{\theta}^2\right] - mga\cos\theta$$

(*) Ec. Lagrange:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0}$$

$$\frac{\partial L}{\partial \theta} = m a^2\omega^2 \sin\theta\cos\theta + mga\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m a^2 \dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m a^2 \ddot{\theta}$$

$$m a^2 \ddot{\theta} - m a^2 \omega^2 \sin\theta\cos\theta - mga\sin\theta = 0$$

$$\boxed{\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta + \frac{g}{a}\right)}$$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow H \text{ se conserva}$$

(i) $\vec{r}$ depende explícitamente de t $\Longrightarrow$ $\boxed{H \neq E}$

$$E = T + V$$

$$\longrightarrow E = \frac{1}{2} m \left[a^2\omega^2\sin^2\theta + a^2\dot{\theta}^2\right] + mga\cos\theta$$

$$\frac{dE}{dt} = \frac{\cancel{1}}{\cancel{2}} m\left[a^2\omega^2 \cancel{2}\sin\theta\cos\theta\,\dot{\theta} + a^2 \cancel{2}\dot{\theta}\ddot{\theta}\right] - mga\sin\theta\,\dot{\theta} \neq 0$$

$\hookrightarrow$ E no se conserva

$$\longrightarrow H = \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L = m a^2\dot{\theta}^2 - \frac{1}{2} m\left[a^2\omega^2\sin^2\theta + a^2\dot{\theta}^2\right] +$$

$$+ mga\cos\theta = \underbrace{\frac{1}{2} m a^2\dot{\theta}^2}_{\geq 0 \text{ (T. cinético)}} - \frac{1}{2} m a^2\omega^2\sin^2\theta + mga\cos\theta = cte$$

$$\boxed{V_{ef}(\theta) = -\frac{1}{2} m a^2\omega^2\sin^2\theta + mga\cos\theta}$$

Movimiento en aquellos θ tales que $H \geq V_{ef}(\theta)$

$$\frac{dV_{ef}}{d\theta} = -ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta$$

Mínimo en $\theta_0 \longrightarrow \left.\frac{dV_{ef}}{d\theta}\right|_{\theta_0} = 0 \longrightarrow m a^2\omega^2\sin\theta_0\cos\theta_0 + mga\sin\theta_0 =$

$a\omega^2\cos\theta_0 + g = 0 \longrightarrow$

$$\cos\theta_0 = -\frac{g}{a\omega^2} \longrightarrow |\cos\theta_0| \leq 1$$

2 Ejs

$$\frac{g/a}{\omega^2} \leq 1 \Rightarrow \underbrace{\omega^2 \geq \frac{g}{a} = \omega_c^2}_{\omega_c \equiv \text{frecuencia crítica}}$$

Para que exista un mínimo en θ_0, $\boxed{\omega \geq \omega_c} = \sqrt{\frac{g}{a}}$

$$\frac{d^2 V_{ef}}{d\theta^2} = -ma^2\omega^2\cos^2\theta + ma^2\omega^2\sin^2\theta - mga\cos\theta$$

$$\left.\frac{d^2 V_{ef}}{d\theta^2}\right|_{\theta_0} = -ma^2\omega^2\left(-\frac{g}{a\omega^2}\right)^2 + ma^2\omega^2\left(1-\left(-\frac{g}{a\omega^2}\right)^2\right) - mga\left(-\frac{g}{a\omega^2}\right) =$$

$$= ma^2\left[\frac{\omega^2 g^2}{a^2\omega^4} + \omega^2 - \frac{g^2\omega^2}{a^2\omega^4} - \frac{g^2}{a^2\omega^2}\right] =$$

(*) $g^2/a^2 = \omega_c^4$

$$= ma^2\left[\frac{\omega_c^4}{\omega^2} + \omega^2 - \frac{\omega_c^4}{\omega^2} - \frac{\omega_c^4}{\omega^2}\right] =$$

$$= \underbrace{ma^2\omega^2}_{\geq 0 \Rightarrow}\underbrace{\left(1-\frac{\omega_c^4}{\omega^4}\right)}_{\geq 0} \geq 0 \text{ para que } \hookrightarrow \theta_0 \text{ (mínimo)}$$

$$\Updownarrow \frac{\omega_c^4}{\omega^4} \leq 1 \Longleftrightarrow \omega_c^4 \leq \omega^4$$

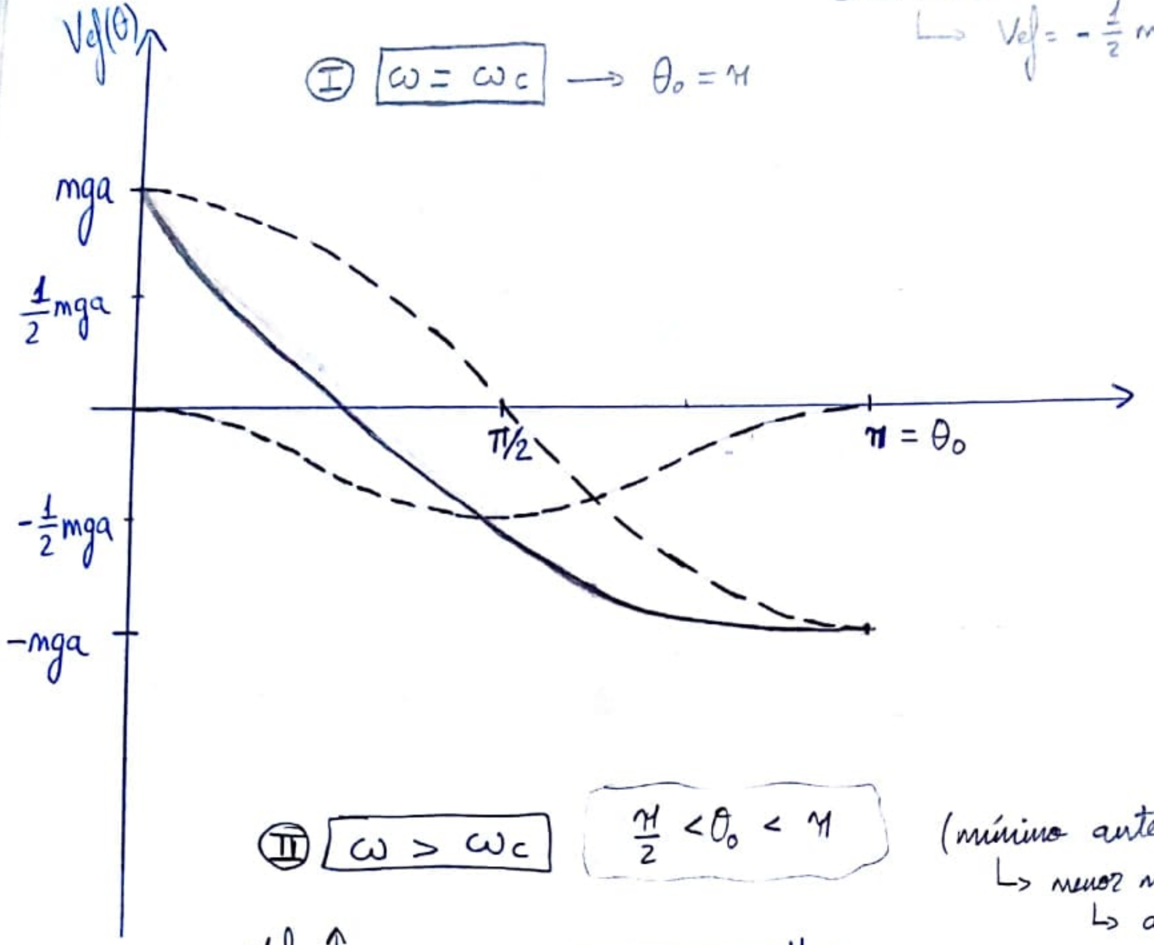
Cierto porque $\omega \geq \omega_c$ es condición para que la primera derivada sea nula (de haber un extremo relativo del V_{ef} en θ_0 tiene que ser un mínimo)

$$\text{(*)}\ \underbrace{\cos\theta_0 = -\frac{\omega_c^2}{\omega^2} < 0}_{\theta_0 > \frac{\pi}{2}\quad (\theta \in [0,\pi])}$$

(*) $\omega = \omega_c$

↳ $V_{ef} = -\frac{1}{2} m a g \sin^2\theta + m g a \cos\theta$

(I) $\boxed{\omega = \omega_c} \longrightarrow \theta_0 = \pi$



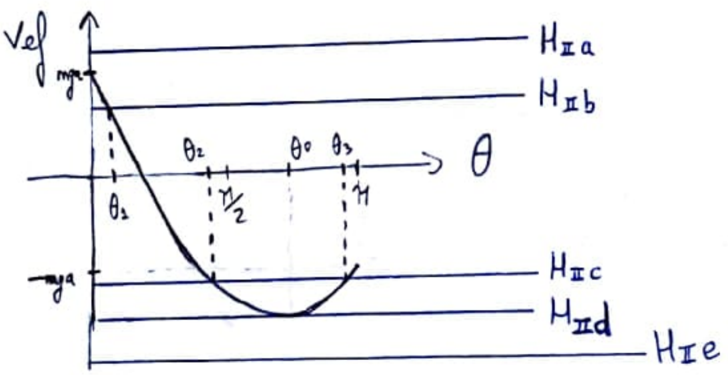
(II) $\boxed{\omega > \omega_c}$ $\boxed{\frac{\pi}{2} < \theta_0 < \pi}$ (mínimo antes)

↳ menor valor

↳ ao ser ω^2 maior,

$-\frac{1}{2} m a^2 \omega^2 \sin^2\theta$

é moito máis negativo

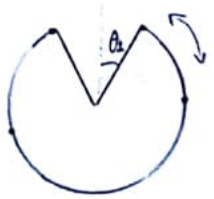


$\omega \geq \omega_c$

(IIa) $H = H_{IIa} > mga \Rightarrow \theta \in [0, \pi]$

(IIb) $H = H_{IIb}$; $-mga < H_{IIb} < mga$

$\theta \in [\theta_1, \pi]$



(*) (I) $\omega = \omega_c$

↓

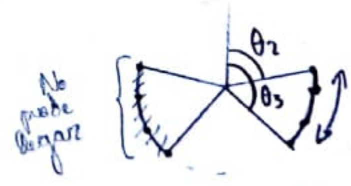
$\theta_3 = \theta_0 =$

(Ib) y (Ic)

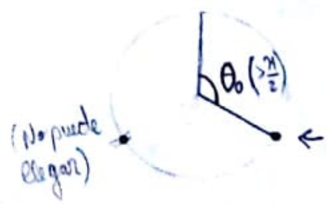
mismo caso

(IIc) $H = H_{IIc}$, $-mga < H_{IIc} < V_{ef}(\theta_0)$

$\theta \in [\theta_2, \theta_3]$

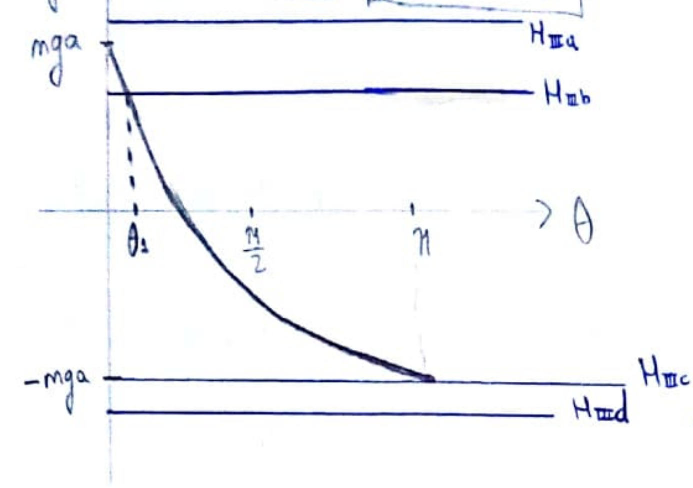


(IId) $H = H_{IId} = V_{ef}(\theta_0)$ $\boxed{\theta = \theta_0}$



(IIe) $H = H_{IIe} < V_{ef}(\theta_0) \rightarrow$ Ningún θ permitido

(III) $\boxed{\omega < \omega_c}$



$\nexists\ \theta_0 \ / \ V_{ef}(\theta_0) \to$ mínimo

(IIIa) $H = H_{IIIa} \geq mga$

$\theta \in [0, \pi]$

(IIIb) $H = H_{IIIb}$

$-mga < H_{IIIb} < mga$

$\theta \in [\theta_1, \pi]$

(IIId) $H_{IIId} < -mga$

Ninguna posición permitida

(IIIc) $H = H_{IIIc} = -mga$

$\boxed{\theta = \pi}$

Tema 3: Oscilaciones lineales

→ Oscilador armónico:

Partícula de masa m sometida a fuerzas en 1D:

(a) $F_{rec} = -kx$, $k > 0$ Ley de Hooke

(b) $F_{roz} = -bv$, $b > 0$ Rozamiento. $(v = \dot{x})$

(c) $F_{arb} = F(t)$ Fuerza externa

Newton: $m\ddot{x} = \sum F \longrightarrow m\ddot{x} = -kx - b\dot{x} + F(t)$

$$\boxed{m\ddot{x} + b\dot{x} + kx = F(t)}$$

(E.D. lineal coef ctes no homogénea de 2º orden)

$\boxed{3.1}$ Oscilador armónico: $b = 0$ $F(t) = 0$

Ec. mov. → $\boxed{m\ddot{x} = -kx}$ $\boxed{\circledast\ \omega_0^2 = \frac{k}{m}}$

$$\ddot{x} = -\frac{k}{m}x$$

$$\ddot{x} = -\omega_0^2 x \qquad \ddot{x} + \omega_0^2 x = 0 \qquad \circledast\ x = e^{\lambda t}$$

$$\lambda^2 + \omega_0^2 = 0$$

$$\boxed{\lambda = \pm i\omega_0^2}$$

$$\boxed{x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}}$$

⊛ Fórmula de Euler: $\boxed{e^{i\alpha} = \cos\alpha + i\sin\alpha}$

$$x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t} = c_1\left[\cos(\omega_0 t) + i\sin(\omega_0 t)\right] +$$

$$+ c_2\left[\cos(\omega_0 t) - i\sin(\omega_0 t)\right] = (c_1+c_2)\cos(\omega_0 t) + i(c_1-c_2)\sin(\omega_0 t) =$$

$$= \alpha\cos(\omega_0 t) + \beta\sin(\omega_0 t) \qquad (\alpha, \beta \in \mathbb{C})$$

$$x(t) = A\left(\frac{\alpha}{A}\cos\omega_0 t + \frac{\beta}{A}\sin\omega_0 t\right) \overset{*}{=}$$

$$= A\left(\sin\delta\cos\omega_0 t + \cos\delta\sin\omega_0 t\right) =$$

$$= A\sin(\omega_0 t + \delta) \overset{*}{=} A\cos(\omega_0 t + \phi)$$

(*) Elegimos un A tal que $\left(\frac{\alpha}{A}\right)^2 + \left(\frac{\beta}{A}\right)^2 = 1$,

$$A = \sqrt{\alpha^2 + \beta^2}$$

de forma que podemos llamar $\frac{\alpha}{A} = \sin\delta$ y $\frac{\beta}{A} = \cos\delta$

Ec. posición:

$$\longrightarrow \boxed{x(t) = A\cos(\omega_0 t + \phi)}$$

$A \equiv$ amplitud

$\omega_0 \equiv \sqrt{\frac{k}{m}} \equiv$ frecuencia natural

$\phi \equiv$ fase inicial

$$\dot{x} = -A\omega_0\sin(\omega_0 t + \phi)$$

$$\ddot{x} = -A\omega_0^2\cos(\omega_0 t + \phi) = -\omega_0^2 x$$

(*) $\sin(\alpha) = \cos\left(\frac{\pi}{2} - \alpha\right)$

$$\sin(\omega_0 t + \delta) = \cos\left(\frac{\pi}{2} - \omega_0 t - \delta\right) =$$

$$= \cos\left(-\omega_0 t + \frac{\pi}{2} - \delta\right) =$$

$$= \cos\left(\omega_0 t + \delta - \frac{\pi}{2}\right) =$$

$$= \cos(\omega_0 t + \phi)$$

$$\boxed{\phi = \delta - \frac{\pi}{2}}$$

$\longrightarrow$ Energías:

- Cinética:

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left(-A\omega_0\sin(\omega_0 t + \phi)\right)^2 = \frac{1}{2}m\omega_0^2 A^2\sin^2(\omega_0 t + \phi)$$

$$\boxed{T = \frac{1}{2}kA^2\sin^2(\omega_0 t + \phi)} \longleftarrow \omega_0^2 = \frac{k}{m}$$

- Potencial:

$$V = \frac{1}{2}kx^2, \qquad \boxed{V = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)}$$

- Total:

$$E = T + V = \frac{1}{2}kA^2\left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)\right]$$

$$\boxed{E = \frac{1}{2}kA^2 = \text{cte}} \longrightarrow \text{La energía se conserva}$$

→ Período : $\omega_0 \tau_0 = 2\pi$, $\boxed{\tau_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}}$

3.2 Oscilador amortiguado : $b \neq 0$, $F(t) = 0$

$$\boxed{m\ddot{x} = -kx - b\dot{x}}$$

$$\boxed{(*)\ \omega_0^2 = \frac{k}{m} \quad \gamma = \frac{b}{2m}}$$

→ coeficiente de amortiguamiento

$$\ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0}$$

$(*)$ $x = e^{\lambda t}$

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$$

$$\left.\begin{aligned} \lambda_1 &= -\gamma + \sqrt{\gamma^2 - \omega_0^2} \\ \lambda_2 &= -\gamma - \sqrt{\gamma^2 - \omega_0^2} \end{aligned}\right\} \lambda = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

↳ Solución general :

$$x(t) = A_1 e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + A_2 e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t} =$$

$$= e^{-\gamma t}\left[A_1 e^{\sqrt{\gamma^2 - \omega_0^2}\,t} + A_2 e^{-\sqrt{\gamma^2 - \omega_0^2}\,t}\right]$$

3 casos :

- I $\boxed{\omega_0^2 > \gamma^2}$ oscilador armónico infraamortiguado
- II $\boxed{\omega_0^2 < \gamma^2}$ oscilador armónico sobreamortiguado
- III $\boxed{\omega_0^2 = \gamma^2}$ oscilador armónico con amortiguamiento crítico

I Infraamortiguado : $\omega_0^2 > \gamma^2$, Def.- $\boxed{\omega_1^2 \equiv \omega_0^2 - \gamma^2} \in \mathbb{R}$

$$\gamma^2 - \omega_0^2 < 0 \implies \sqrt{\gamma^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \gamma^2} = i\,\omega_1$$

$$x(t) = e^{-\gamma t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right] = A e^{-\gamma t}\left[\cos(\omega_1 t - \delta)\right]$$

$$\boxed{x(t) = A\,e^{-\gamma t} \cdot \cos(\omega_1 t - \delta)}$$

$\omega_1 \equiv$ frecuencia angular del movimiento

$\underbrace{\omega_1^2 \equiv \omega_0^2 - \gamma^2}$ $(\gamma \neq 0$ (hay roz.))

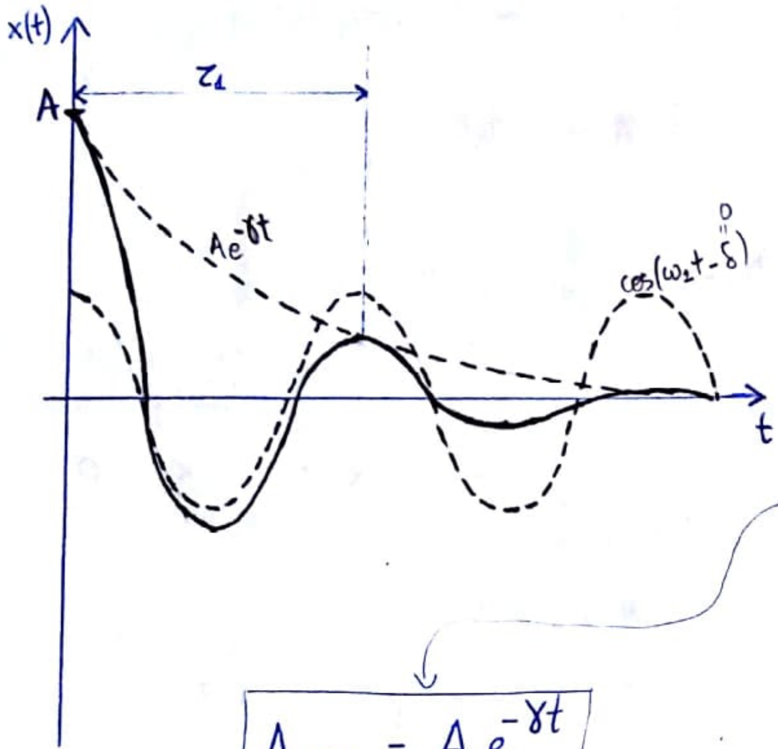
$\Downarrow$

$\omega_1^2 < \omega_0^2$, $\underbrace{\boxed{\omega_1 < \omega_0}}$

$\Downarrow$

$\boxed{\tau_0 < \tau_1} \Leftarrow \dfrac{2\pi}{\tau_1} < \dfrac{2\pi}{\tau_0}$

$\hookrightarrow$ mayor período que el oscilador armónico



$\underbrace{e^{-\gamma t} A \equiv \text{amplitudes máximas (cuando } \cos(\,) = 1)}$

la amplitud va disminuyendo con el transcurso del tiempo

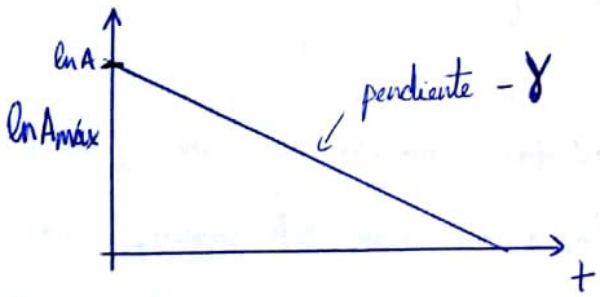
$\underbrace{\left(\gamma \equiv \dfrac{b}{2m} > 0\right)}$

$\lim_{t \to \infty} x(t) = 0$ (Se para)

$$\boxed{A_{max} = A\,e^{-\gamma t}}$$

la amplitud máxima decae exponencialmente con el tiempo

$$\boxed{\ln(A_{max})} = \ln(A\,e^{-\gamma t}) = \ln(A) + \ln(e^{-\gamma t}) = \boxed{\ln A - \gamma t}$$

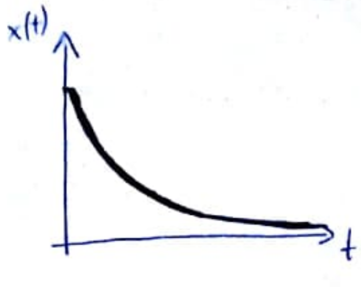


(II) Sobreamortiguado : $\gamma^2 > \omega_0^2$, Def.- $\boxed{\omega_2 \equiv \sqrt{\gamma^2 - \omega_0^2}} \in \mathbb{R}$

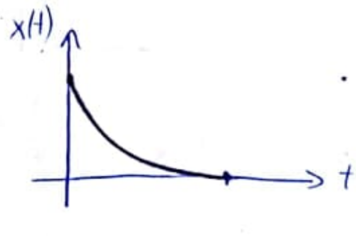
$$\boxed{x(t)} = e^{-\gamma t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right] =$$

$$= A_1 e^{(-\gamma + \omega_2)t} + A_2 e^{(-\gamma - \omega_2)t} = \boxed{A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}}$$

Def.- $\begin{cases} -\gamma_1 \equiv -\gamma + \sqrt{\gamma^2 - \omega_0^2} \\ -\gamma_2 \equiv -\gamma - \sqrt{\gamma^2 - \omega_0^2} \end{cases}$



No hay oscilación (se para)



III Amortiguamiento crítico: $\gamma^2 = \omega_0^2 \Rightarrow$ Def.- $\boxed{\gamma_c = \gamma}$

crítico

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \qquad \boxed{\circledast \; x = e^{\lambda t}}$$

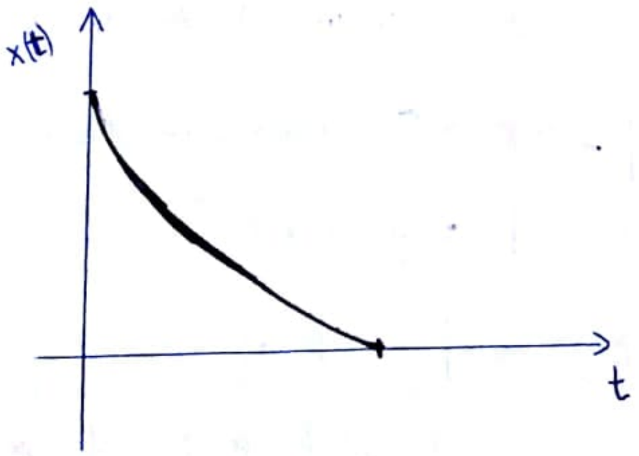
$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0 \qquad \gamma^2 = \omega_0^2$$

$$\lambda = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \longrightarrow \boxed{\circledast 2}$$

$$\boxed{x(t) = A e^{-\gamma t} + Bt e^{-\gamma t}}$$

$$\boxed{x(t) = (A + Bt)\, e^{-\gamma t}} \rightarrow$$

Alcanza más rápido la posición de equilibrio (sin oscilar)



3.3 Oscilador amortiguado y forzado: $b \neq 0, \; F(t) \neq 0$

$$\boxed{m\ddot{x} + kx + b\dot{x} = F(t)} \qquad \circledast \text{Def.-} \boxed{\omega_0^2 = \frac{k}{m}, \; \gamma = \frac{b}{2m}}$$

$$\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = F(t)/m \longrightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t)\frac{1}{m}$$

∘ Resolvemos el caso $\boxed{F(t) = F_0 \cos(\omega t)}$

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

$$\boxed{x(t) = x_h(t) + x_p(t)}$$

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0 \qquad \boxed{⊛\ x = e^{\lambda t}}$$

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$$

$$\lambda = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\boxed{x_h(t) = \left[A_1 \exp\left(\sqrt{\gamma^2-\omega_0^2}\, t\right) + A_2 \exp\left(-\sqrt{\gamma^2-\omega_0^2}\, t\right)\right] e^{-\gamma t}}$$

⊛ probamos una solución particular: $\boxed{x_p(t) = D \cos(\omega t - \delta)}$

$$\dot{x}_p(t) = -D\omega \sin(\omega t - \delta) \qquad \ddot{x}_p(t) = -D\omega^2 \cos(\omega t - \delta)$$

⊛ Def - $\boxed{\frac{F_0}{m} \equiv f_0}$

$$\ddot{x}_p + 2\gamma \dot{x}_p + \omega_0^2 x_p = \frac{F_0}{m} \cos(\omega t) = f_0 \cos(\omega t)$$

$$-D\omega^2 \cos(\omega t - \delta) + 2\gamma\left(-D\omega \sin(\omega t - \delta)\right) + \omega_0^2 D \cos(\omega t - \delta) = f_0 \cos\omega t$$

$$-D\omega^2 [\cos\omega t \cos\delta + \sin\omega t \sin\delta] - 2\gamma\omega D [\sin\omega t \cos\delta - \cos\omega t \sin\delta] +$$
$$+ \omega_0^2 D [\cos\omega t \cos\delta + \sin\omega t \sin\delta] = f_0 \cos\omega t$$

$$[-D\omega^2 \cos\delta + 2\gamma\omega D \sin\delta + D\omega_0^2 \cos\delta] \cos\omega t +$$
$$+ [-D\omega^2 \sin\delta - 2\gamma\omega D \cos\delta + D\omega_0^2 \sin\delta] \sin\omega t = f_0 \cos\omega t$$

$\forall t$

$\Downarrow$

$$\left.\begin{aligned} -D\omega^2 \cos\delta + 2\gamma\omega D \sin\delta + D\omega_0^2 \cos\delta &= f_0 \\ -D\omega^2 \sin\delta - 2\gamma\omega D \cos\delta + D\omega_0^2 \sin\delta &= 0 \end{aligned}\right\}$$

Sistema 2 ecuaciones, 2 incógnitas (D, δ)

$$f_0 - D(\omega_0^2 - \omega^2)\cos\delta - 2\gamma\omega D \sin\delta = 0 \quad (1)$$

$$-D(\omega_0^2 - \omega^2)\sin\delta + 2\gamma\omega D\cos\delta = 0 \quad (2)$$

De (2) $\longrightarrow$ $\tan\delta = \dfrac{\sin\delta}{\cos\delta} = \dfrac{2\gamma\omega D}{+D(\omega_0^2-\omega^2)} = \dfrac{2\gamma\omega}{\omega_0^2-\omega^2}$

$$\cos\delta = \frac{1}{\sqrt{1+\tan^2\delta}} = \frac{1}{\sqrt{1+\dfrac{4\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2}}} = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$$\sin\delta = \tan\delta\cdot\cos\delta = \frac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$\longrightarrow$ Sustituimos en (1):

$$f_0 - D(\omega_0^2-\omega^2)\cos\delta - D\,2\gamma\omega\sin\delta = 0$$

$$D = \frac{f_0}{(\omega_0^2-\omega^2)\cos\delta + 2\gamma\omega\sin\delta} =$$

$$= \frac{f_0}{(\omega_0^2-\omega^2)\dfrac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} + 2\gamma\omega\dfrac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}} =$$

$$= \frac{f_0\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

$$\boxed{D = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}} \longrightarrow x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)+4\gamma^2\omega^2}}\cos(\omega t-\delta)$$

$$\boxed{x(t) = e^{-\gamma t}\left[A_1 e^{\sqrt{\gamma^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right] + \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\cos(\omega t-\delta)}$$

Inframortiguado y forzado: $\omega_0^2 > \gamma^2$ $\quad F(t) = F_0 \cos \omega t$

$$\boxed{x(t) = \underbrace{A e^{-\gamma t}\cos(\omega_1 t + \theta)}_{\text{Término transitorio}} + \underbrace{\frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\cos(\omega t - \delta)}_{\text{Término estacionario}}}$$

Término transitorio

→ Oscila con frecuencia ω_1

$\omega_1^2 \equiv \omega_0^2 - \gamma^2$

→ Para $\left(t \gg \frac{1}{\gamma}\right) \equiv t_\gamma \ggg 1$ tiende a 0

→ Depende de las c.i. (A, θ)

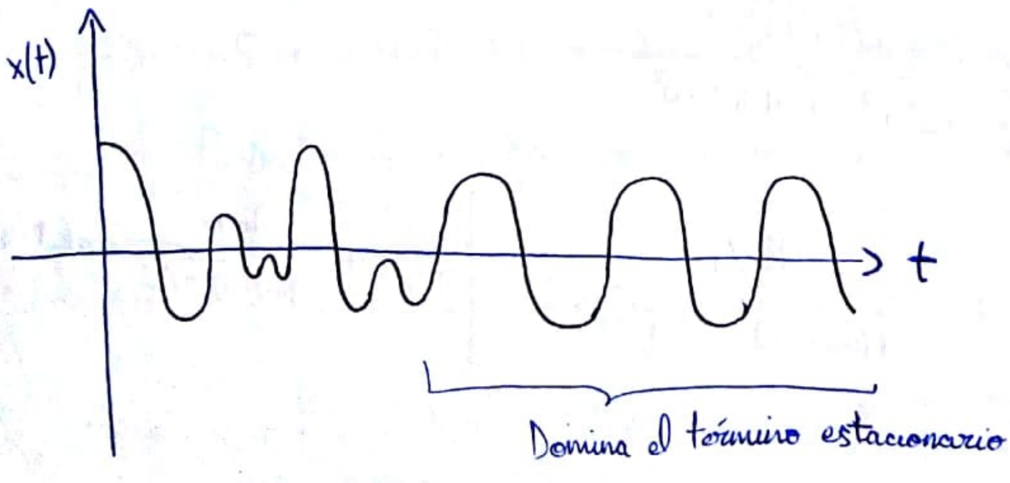
Término estacionario

→ Domina cuando $t \gg t_\gamma = \frac{1}{\gamma}$ (cuando desaparece el transitorio)

→ No depende de las c.i. (δ solo depende del oscilador)

→ La partícula oscila con frecuencia ω, la frecuencia de la fuerza externa, pero desfasada δ.

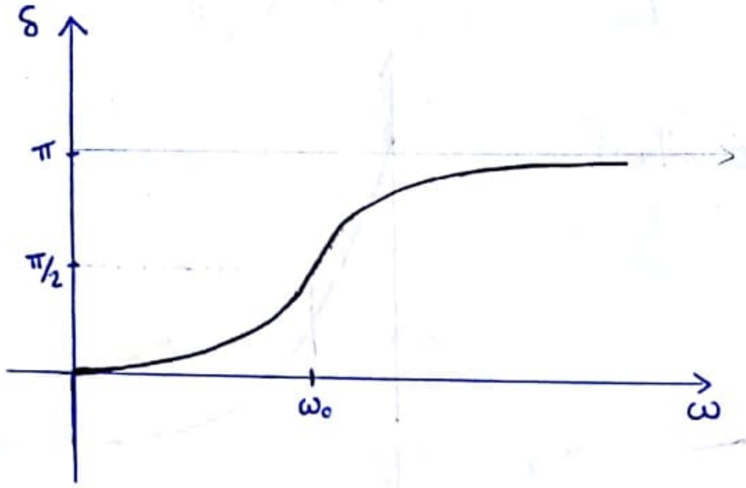
$$\begin{cases} \omega_0^2 = \dfrac{k}{m}, & \omega_0 \equiv \text{frecuencia natural del oscilador} \\ & \omega \equiv \text{frecuencia de la fuerza externa} \end{cases}$$



$$\tan\delta = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

$$\begin{cases} \omega = 0 \Rightarrow \tan\delta = 0 \Rightarrow \delta = 0 \\ \omega = \omega_0 \Rightarrow \tan\delta \to \infty \Rightarrow \delta \to \frac{\pi}{2} \\ \omega \to \infty \Rightarrow \tan\delta \to 0^- \Rightarrow \delta = \pi \end{cases}$$

δ depende de la frecuencia ω



$\delta \equiv$ desfase entre la fuerza $F(t)$ y la oscilación $x(t)$.

→ Resonancia

La amplitud $D = \dfrac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$ del término estacionario depende de ω. La frecuencia de resonancia ω_R es aquella para la que la amplitud $D(\omega)$ es máxima.

$$\frac{dD}{d\omega} = \frac{F_0}{m}\left(-\frac{1}{2}\right)\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{-\frac{3}{2}}\cdot\left[2(\omega_0^2-\omega^2)(-2\omega)+4\gamma^2\cdot 2\omega\right]$$

$$\boxed{\left.\frac{dD}{d\omega}\right|_{\omega_R} = 0}\quad , \quad \underbrace{\underbrace{\frac{F_0}{m}\left(-\frac{1}{2}\right)\left[(\omega_0^2-\omega_R^2)^2+4\gamma^2\omega_R^2\right]^{-3/2}}_{\neq 0}\left[-4\omega_R(\omega_0^2-\omega_R^2)+8\gamma^2\omega_R\right] = 0}$$

$$\boxed{\omega_R \neq 0}$$

$$\Downarrow$$

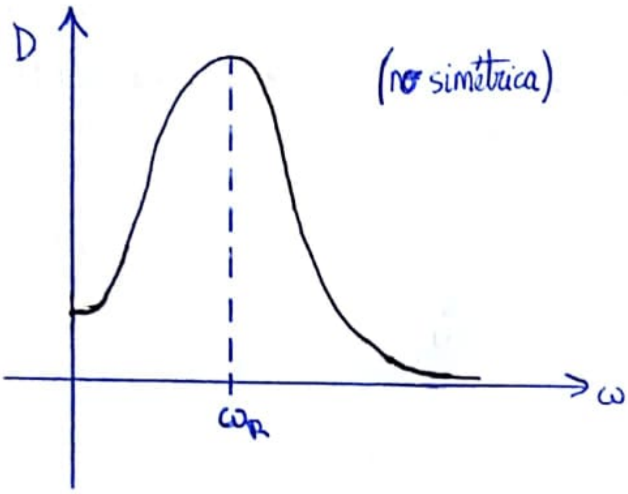
$$-\omega_R(\omega_0^2-\omega_R^2)+2\gamma^2\omega_R = 0$$

$$2\gamma^2 = \omega_0^2-\omega_R^2$$

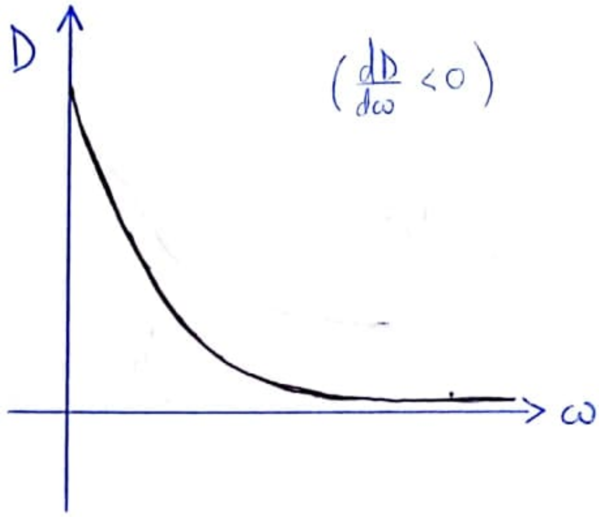
$$\boxed{\omega_R^2 = \omega_0^2 - 2\gamma^2}$$

$$\boxed{\exists\, \omega_R \iff \omega_0^2 > 2\gamma^2}$$

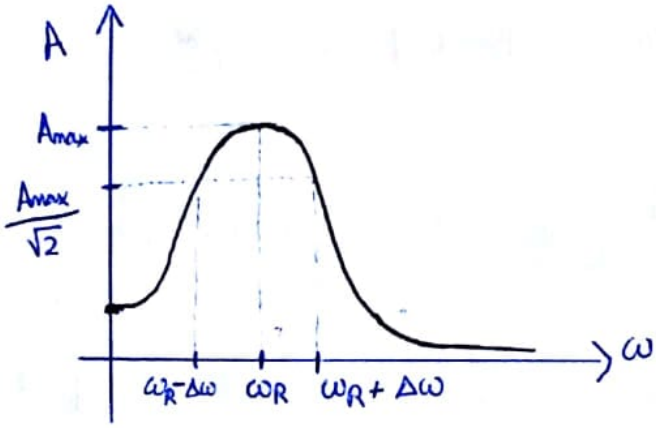
- $\omega_0^2 > 2\gamma^2$: $(\exists\, \omega_R)$



- $\omega_0^2 < 2\gamma^2$: $(\nexists\, \omega_R)$



→ Anchura de la curva de resonancia $(2\Delta\omega)$



Semianchura de la curva:

$$\boxed{\Delta\omega = \gamma_\omega = \gamma\sqrt{1 + \frac{\gamma^2}{\omega_R^2}}}$$

↳ Demostración:

$$A(\omega_R + \Delta\omega) = A(\omega_R - \Delta\omega) = \frac{A_{max}}{\sqrt{2}}$$

$$A(\omega_R + \Delta\omega) = \frac{F_0/m}{\sqrt{\left[\omega_0^2 - (\omega_R + \Delta\omega)^2\right]^2 + 4\gamma^2(\omega_R + \Delta\omega)^2}}$$

$$\left[\omega_0^2 - (\omega_R + \Delta\omega)^2\right]^2 + 4\gamma^2(\omega_R + \Delta\omega)^2 =$$

$$= \left[\omega_0^2 - \omega_R^2 - \Delta\omega^2 - 2\omega_R\Delta\omega\right]^2 + 4\gamma^2\left[\omega_R^2 + \Delta\omega^2 + 2\ldots\right.$$

$$= \left[(\omega_0^2 - \omega_R^2) + (-\Delta\omega^2) + (-2\omega_R\Delta\omega)\right]^2 + 4\gamma^2\left[\omega_R^2 + \Delta\omega^2 + 2\omega_R\ldots\right.$$

$$= (\omega_0^2 - \omega_R^2)^2 + \Delta\omega^4 + 4\omega_R^2\Delta\omega^2 - 4(\omega_R\Delta\omega)(\omega_0^2 - \omega_R^2)$$

$$- 2\Delta\omega^2(\omega_0^2 - \omega_A^2) + 4\omega_R\Delta\omega^3 + 4\gamma^2\omega_R^2 + 4\gamma^2\Delta\omega^2 + 8\gamma^2\omega_R\Delta\ldots$$

⊛ Para γ pequeños:

$$\underbrace{\omega_R^2 = \omega_0^2 - 2\gamma}$$

$$\gamma \to 0 \Rightarrow \underbrace{\omega_R^2 \simeq \omega_0^2}$$

$$\omega_R \simeq \omega_0$$

$$\underbrace{\Delta\omega = \gamma_\omega = \gamma\left(1 + \frac{\gamma^2}{\omega_R^2}\right)^{1/2}}$$

$$\gamma \to 0 \Rightarrow \boxed{\Delta\omega = \gamma_\omega \simeq \gamma}$$

$$\hookrightarrow \boxed{2\Delta\omega \simeq 2\gamma}$$

$$\approx (\omega_0^2-\omega_R^2)^2 + 4\omega_R^2\Delta\omega^2 - 4\omega_R\Delta\omega(\omega_0^2-\omega_R^2) - 2(\omega_0^2-\omega_R^2)\Delta\omega^2 +$$

$$+4\omega_R^2\gamma^2 + 8\gamma^2\omega_R\Delta\omega + 4\gamma^2\Delta\omega^2 = \circledast \quad \boxed{\begin{array}{l}\omega_R^2 = \omega_0^2 - 2\gamma^2\\ \omega_0^2 - \omega_R^2 = 2\gamma^2\end{array}}$$

despreciamos términos $\Delta\omega^3, \Delta\omega^4 \ldots$

$$= (2\gamma^2)^2 + 4\omega_R^2\Delta\omega^2 - 4\omega_R\Delta\omega(2\gamma^2) - 2(2\gamma^2)\Delta\omega^2 +$$

$$+4\omega_R^2\gamma^2 + 8\gamma^2\omega_R\Delta\omega + 4\gamma^2\Delta\omega^2 = 4\gamma^4 + 4\omega_R^2(\gamma^2+\Delta\omega^2) =$$

$$= 4\omega_R^2\left[\Delta\omega^2 + \gamma^2\left(1+\frac{\gamma^2}{\omega_R^2}\right)\right] = 4\omega_R^2\left[\Delta\omega^2 + \underbrace{\left(\gamma\sqrt{1+\frac{\gamma^2}{\omega_R^2}}\right)^2}_{\text{Def.} - \gamma_\omega}\right] =$$

$$= 4\omega_R^2\left[\Delta\omega^2 + \gamma_\omega^2\right]$$

$$A(\omega_R+\Delta\omega) \approx \frac{F_0/m}{\sqrt{4\omega_R^2(\Delta\omega^2+\gamma_\omega^2)}} = \frac{F_0/m}{2\omega_R\sqrt{\Delta\omega^2+\gamma_\omega^2}}$$

$$\underset{(\Delta\omega=0)}{A_{max}} = \frac{F_0/m}{2\omega_R\gamma_\omega} = \frac{F_0}{2\omega_R m\gamma_\omega}$$

$$\boxed{A(\omega_R+\Delta\omega) = \frac{A_{max}}{\sqrt{2}}}$$

$$\frac{F_0}{2m\omega_R\sqrt{\Delta\omega^2+\gamma_\omega^2}} = \frac{F_0}{2\sqrt{2}\,\omega_R m\gamma_\omega}\ , \qquad \sqrt{2}\,\gamma_\omega = \sqrt{\Delta\omega^2+\gamma_\omega^2}$$

$$2\gamma_\omega^2 = \Delta\omega^2 + \gamma_\omega^2$$

$$\Delta\omega^2 = \gamma_\omega^2$$

$$\boxed{\Delta\omega = \gamma_\omega} \quad q.e.d.$$

→ Principio de superposición:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t)$$

Sea el operador diferencial lineal: $L \equiv \left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right)$

$$\begin{cases} L\, x_1(t) = F_1(t) \\ L\, x_2(t) = F_2(t) \end{cases}$$

Si $F(t) = \sum_{n=0}^{\infty} \alpha_n F_n$, entonces la solución $x(t)$ de la ecuación $L\, x(t) = F(t)$ es $x(t) = \sum_{n=0}^{\infty} \alpha_n x_n(t)$, donde $x_n(t)$ verifica $L\, x_n(t) = F_n(t)$.

⊛ Teorema de Fourier :

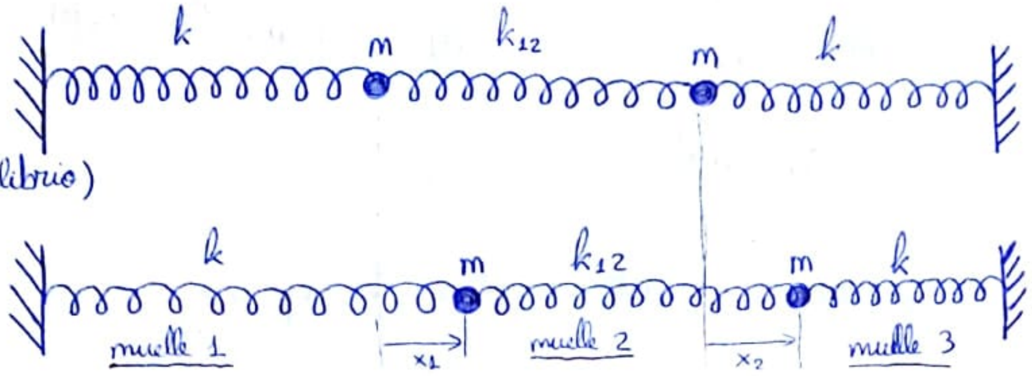
Si $F(t)$ es una función periódica con periodo τ $\left(F(t) = F(t+\tau)\right)$ entonces $F(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$.

$$a_n = \frac{2}{\tau} \int_0^{\tau} F(t) \cos(n\omega t)\, dt \qquad a_0 = \frac{2}{\tau} \int_0^{\tau} F(t)\, dt$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} F(t) \sin(n\omega t)\, dt$$

Osciladores acoplados:

Ejemplo



$\boxed{V=0}$ → Tomamos el plano horizontal a la altura del sistema como origen de potencial gravitatorio

desplazamiento de las masas con respecto de sus posiciones de equilibrio.

Lagrangiano del sistema: $L = T - V$

$$L = \left(\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2\right) - \left(\underbrace{\frac{1}{2} k x_1^2}_{\text{muelle 1}} + \underbrace{\frac{1}{2} k x_2^2}_{\text{muelle 3}} + \underbrace{\frac{1}{2} k_{12} (x_2 - x_1)^2}_{\text{muelle 2}}\right)$$

Ecuaciones de Lagrange: $\boxed{\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right) - \dfrac{\partial L}{\partial q_j} = 0}$

x_1

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) &= \frac{d}{dt}\left(m \dot{x}_1\right) = m \ddot{x}_1 \\ \frac{\partial L}{\partial x_1} &= -k x_1 + k_{12}(x_2 - x_1). \end{aligned}\right\} \quad \boxed{m \ddot{x}_1 = -k x_1 - k_{12}(x_1 - x_2)} \quad \boxed{1}$$

$\boxed{\text{Ecuaciones acopladas}}$

x_2

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) &= \frac{d}{dt}\left(m \dot{x}_2\right) = m \ddot{x}_2 \\ \frac{\partial L}{\partial x_2} &= -k x_2 - k_{12}(x_2 - x_1) \end{aligned}\right\} \quad \boxed{m \ddot{x}_2 = -k x_2 - k_{12}(x_2 - x_1)} \quad \boxed{2}$$

Solución oscilatoria:

$$x_1(t) = B_1 e^{i\omega t}, \quad \dot{x}_1(t) = i B_1 \omega e^{i\omega t}, \quad \ddot{x}_1(t) = -B_1 \omega^2 e^{i\omega t}$$

$$x_2(t) = B_2 e^{i\omega t}, \quad \dot{x}_2(t) = i B_2 \omega e^{i\omega t}, \quad \ddot{x}_2(t) = -B_2 \omega^2 e^{i\omega t}$$

¿ B_1, B_2, ω ?

Sustituimos en las ecuaciones de Lagrange:

$$-m B_1 \omega^2 e^{i\omega t} = -k B_1 e^{i\omega t} + k_{12} (B_2 - B_1) e^{i\omega t}$$

$$-m B_1 \omega^2 = -k B_1 + k_{12} (B_2 - B_1)$$

$$\boxed{(k + k_{12} - m\omega^2) B_1 - k_{12} B_2 = 0} \quad \text{(a)}$$

$$-m B_2 \omega^2 e^{i\omega t} = -k B_2 e^{i\omega t} - k_{12} (B_2 - B_1) e^{i\omega t}$$

$$-m B_2 \omega^2 = -k B_2 - k_{12} (B_2 - B_1)$$

$$\boxed{-k_{12} B_1 + (k + k_{12} - m\omega^2) B_2 = 0} \quad \text{(b)}$$

Para que exista una solución no trivial para B_1 y B_2, el sistema de ecuaciones lineales (a), (b) tiene que ser compatible indeterminado:

$$\exists \text{ sol } (B_1, B_2) \neq (0,0) \iff \begin{vmatrix} k + k_{12} - m\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - m\omega^2 \end{vmatrix} = 0$$

$$(k + k_{12} - m\omega^2)^2 - k_{12}^2 = 0, \quad (k + k_{12} - m\omega^2)^2 = k_{12}^2$$

$$k + k_{12} - m\omega^2 = \pm k_{12}$$

$$m\omega^2 = k + k_{12} \mp k_{12}$$

$$\omega^2 = \frac{k + k_{12} \mp k_{12}}{m} \longrightarrow \omega^2 = \begin{cases} = \dfrac{k}{m} \\ = \dfrac{k+2k_{12}}{m} \end{cases}$$

$$\boxed{\omega = \pm\sqrt{\frac{k + k_{12} \mp k_{12}}{m}}} \longrightarrow \text{4 soluciones}$$

$$\begin{cases} \omega_1 = \sqrt{\dfrac{k}{m}} = \omega_S \qquad \omega_3 = \sqrt{\dfrac{k+k_{12}\cdot 2}{m}} = \omega_A \\ \omega_2 = -\sqrt{\dfrac{k}{m}} = -\omega_S \qquad \omega_4 = -\sqrt{\dfrac{k+2k_{12}}{m}} = -\omega_A \end{cases}$$

$$\boxed{\begin{aligned} x_1(t) &= B_{1S}^+ e^{i\omega_S t} + B_{1S}^- e^{-i\omega_S t} + B_{1A}^+ e^{i\omega_A t} + B_{1A}^- e^{-i\omega_A t} \\ x_2(t) &= B_{2S}^+ e^{i\omega_S t} + B_{2S}^- e^{-i\omega_S t} + B_{2A}^+ e^{i\omega_A t} + B_{2A}^- e^{-i\omega_A t} \end{aligned}}$$

No todas las B son independientes $\longrightarrow$ 2 ecuaciones diferenciales de orden 2 $\longrightarrow$ 4 ctes indeterminadas (no 8)

$$\boxed{\begin{matrix} x_1(0) \\ x_2(0) \\ \dot{x}_1(0) \\ \dot{x}_2(0) \end{matrix}}$$

$\longrightarrow \boxed{\omega = \omega_S}$

$$\begin{cases} x_1(t) = B_{1S}^+ e^{i\omega_S t} \\ x_2(t) = B_{2S}^+ e^{i\omega_S t} \end{cases} \xrightarrow[\text{Lagrange}]{\text{Ecs.}} \begin{cases} (k + k_{12} - m\omega_S^2)\, B_{1S}^+ = k_{12} B_{2S}^+ \\ (k + k_{12} - m\omega_S^2)\, B_{2S}^+ = k_{12} B_{1S}^+ \end{cases}$$

$$\hookrightarrow \left(k + k_{12} - m\frac{k}{m}\right) B_{1S}^+ = k_{12} B_{2S}^+$$

$$k_{12} B_{1S}^+ = k_{12} B_{2S}^+$$

$$\boxed{B_{1S}^+ = B_{2S}^+} = B_S^+$$

$\longrightarrow \boxed{\omega = -\omega_S}$

$$\begin{cases} x_1(t) = B_{1S}^- e^{-i\omega_S t} \\ x_2(t) = B_{2S}^- e^{-i\omega_S t} \end{cases} \quad \begin{cases} \dot{x}_1(t) = -i\omega_S B_{1S}^- e^{-i\omega_S t} \\ \dot{x}_2(t) = -i\omega_S B_{2S}^- e^{-i\omega_S t} \end{cases}$$

$$\begin{cases} \ddot{x}_1(t) = -\omega_s^2 B_{1s}^- e^{-i\omega_s t} \\ \ddot{x}_2(t) = -\omega_s^2 B_{2s}^- e^{-i\omega_s t} \end{cases} \xrightarrow[\text{L.}]{\text{Ecs.}} (k + k_{12} - m\omega_s^2) B_{1s}^- = k_{12} B_{2s}^-$$

$$(k + k_{12} - m\frac{k}{m}) B_{1s}^- = k_{12} B_{2s}^-$$

$$k_{12} B_{1s}^- = k_{12} = B_{2s}^-$$

$$\boxed{B_{1s}^- = B_{2s}^-} = B_s^-$$

$\rightsquigarrow$ $\boxed{\omega = \omega_A}$

$$\begin{cases} x_1(t) = B_{1A}^+ e^{i\omega_A t} \\ x_2(t) = B_{2A}^+ e^{i\omega_A t} \end{cases} \xrightarrow[\text{Lagrange}]{\text{Ecs.}} (k + k_{12} - m\omega_A^2) B_{1A}^+ = k_{12} B_{2A}^+$$

$$\left(k + k_{12} - m\frac{k+2k_{12}}{m}\right) B_{1A}^+ = k_{12} B_{2A}^+$$

$$(-k_{12}) B_{1A}^+ = k_{12} B_{2A}^+$$

$$\boxed{-B_{1A}^+ = B_{2A}^+} = B_A^+$$

$\rightsquigarrow$ $\boxed{\omega = -\omega_A}$

$$\begin{cases} x_1(t) = B_{1A}^- e^{-i\omega_A t} \\ x_2(t) = B_{2A}^- e^{-i\omega_A t} \end{cases} \xrightarrow[\text{Lagrange}]{\text{Ecs.}} (k + k_{12} - m\omega_A^2) B_{1A}^- = k_{12} B_{2A}^-$$

$$\left(k + k_{12} - m\frac{k+2k_{12}}{m}\right) B_{1A}^- = k_{12} B_{2A}^-$$

$$(-k_{12}) B_{1A}^- = k_{12} B_{2A}^-$$

$$\boxed{-B_{1A}^- = B_{2A}^-} = B_A^-$$

Así logramos reducir a 4 el número de ctes arbitrarias que aparecen en las expresiones:

$$x_1(t) = \underline{B_s^+ e^{i\omega_s t} + B_s^- e^{-i\omega_s t}} + \underline{B_A^+ e^{i\omega_A t} + B_A^- e^{-i\omega_A t}}$$

$$x_2(t) = \underline{B_s^+ e^{i\omega_s t} + B_s^- e^{-i\omega_s t}} - \underline{\left(B_A^+ e^{i\omega_A t} + B_A^- e^{-i\omega_A t}\right)}$$

Sol:

$$\boxed{\begin{aligned} x_1(t) &= A_s \cos(\omega_s t - \delta_s) + A_A \cos(\omega_A t - \delta_A) \\ x_2(t) &= A_s \cos(\omega_s t - \delta_s) - A_A \cos(\omega_A t - \delta_A) \end{aligned}} \quad \boxed{\begin{aligned} \omega_s &= \sqrt{\tfrac{k}{m}} \\ \omega_A &= \sqrt{\tfrac{k+2k_{12}}{m}} \end{aligned}}$$

Cualquier solución $x_1(t)$, $x_2(t)$ es combinación lineal de $\cos(\omega_s t - \delta_s)$ y $\cos(\omega_A t - \delta_A)$, soluciones de un oscilador armónico simple con frecuencias $\omega_s = \sqrt{\frac{k}{m}}$ y $\omega_A = \sqrt{\frac{k+2k_{12}}{m}}$

→ Desacoplemos las ecuaciones:

Def. $\begin{cases} \eta_1(t) \equiv x_1(t) - x_2(t) \\ \eta_2(t) \equiv x_1(t) + x_2(t) \end{cases}$ → $\boxed{\begin{aligned} x_1(t) &= \tfrac{1}{2}(\eta_1(t) + \eta_2(t)) \\ x_2(t) &= \tfrac{1}{2}(\eta_2(t) - \eta_1(t)) \end{aligned}}$

↓

coordenadas normales

$$\begin{cases} \ddot{x}_1 = \frac{1}{2}(\ddot{\eta}_1 + \ddot{\eta}_2) \\ \ddot{x}_2 = \frac{1}{2}(\ddot{\eta}_2 - \ddot{\eta}_1) \end{cases}$$

$$\frac{1}{2} m (\ddot{\eta}_1 + \ddot{\eta}_2) = -\frac{1}{2} k (\eta_1 + \eta_2) - \frac{1}{2} k_{12} (\eta_1 + \eta_2 - \eta_2 + \eta_1)$$

$$\boxed{m(\ddot{\eta}_1 + \ddot{\eta}_2) = -k(\eta_1 + \eta_2) - 2k_{12}\eta_1} \quad (a)$$

$$\frac{1}{2} m (\ddot{\eta}_2 - \ddot{\eta}_1) = -\frac{1}{2} k (\eta_2 - \eta_1) - \frac{1}{2} k_{12} (\eta_2 - \eta_1 - \eta_1 - \eta_2)$$

$$\boxed{m(\ddot{\eta}_2 - \ddot{\eta}_1) = -k(\eta_2 - \eta_1) + 2k_{12}\eta_1} \quad (b)$$

$\boxed{(a) + (b)} \Rightarrow \quad 2m\ddot{\eta}_2 = -2k\eta_2 \quad , \quad m\ddot{\eta}_2 = -k\eta_2$

$$\boxed{\ddot{\eta}_2 = -\frac{k}{m}\eta_2}$$

oscilador armónico

$$\boxed{\omega_s^2 = \frac{k}{m}}$$

$$\boxed{(a)-(b)} \Longrightarrow \quad 2m\ddot{\eta}_1 = -2(k+2k_{12})\,\eta_1$$

$$m\ddot{\eta}_1 = -(k+2k_{12})\,\eta_1$$

$$\boxed{\ddot{\eta}_1 = -\frac{k+2k_{12}}{m}\,\eta_1}$$

Oscilador armónico

$$\boxed{\omega_A^2 = \frac{k+2k_{12}}{m}}$$

□ $\eta_1(t)$ oscilador armónico $\omega_A = \sqrt{\frac{k+2k_{12}}{m}}$

$$\eta_1(t) = C_1^+ e^{i\omega_A t} + C_1^- e^{-i\omega_A t} = A_A \cos(\omega_A t - \delta_A)$$

□ $\eta_2(t)$ oscilador armónico $\omega_S = \sqrt{\frac{k}{m}}$

$$\eta_2(t) = C_2^+ e^{i\omega_S t} + C_2^- e^{-i\omega_A t} = A_S \cos(\omega_S t - \delta_S)$$

⊛ Cualquier solución es combinación lineal de η_1 y η_2

(p.e. $x_1(t)$, $x_2(t)$) $\begin{cases} x_1(t) = (\eta_1 + \eta_2)\frac{1}{2} \\ x_2(t) = (\eta_2 - \eta_1)\frac{1}{2} \end{cases}$

→ Modos normales de oscilación:

Escogiendo las condiciones iniciales del sistema siempre podemos lograr que sólo se manifieste una de las coordenadas normales, siendo la otra 0 $\forall t$.

$\boxed{C.i}$ $\boxed{A}$ Si $\boxed{\begin{array}{l} x_1(0) = x_2(0) \\ \dot{x}_1(0) = \dot{x}_2(0) = 0 \end{array}}$ $\Rightarrow \left[\begin{array}{l} \eta_1(0) = 0 \\ \dot{\eta}_1(0) = 0, \ \dot{\eta}_2(0) = 0 \end{array}\right] \Rightarrow$

$$\Rightarrow \left\{\begin{array}{l} \eta_1(0) = 0 \Rightarrow c_1^+ + c_1^- = 0 \\ \dot{\eta}_1(0) = 0 \Rightarrow c_1^+ \omega_A - c_1^- \omega_A = 0, \ c_1^+ - c_1^- = 0 \end{array}\right] \Rightarrow \left|\begin{array}{l} c_1^+ = 0 \\ c_1^- = 0 \end{array}\right|$$

$$\Downarrow$$

$$\underbrace{\boxed{\begin{array}{c} \eta_2(t) \text{ oscila} \\ \text{con } \omega_S = \sqrt{\frac{k}{m}} \end{array}} \Leftarrow \boxed{\begin{array}{c} \eta_1(t) = 0 \\ \forall t \end{array}}}_{\text{Modo normal simétrico}}$$

$\boxed{B}$ Si $\boxed{\begin{array}{l} x_1(0) = -x_2(0) \\ \dot{x}_1(0) = -\dot{x}_2(0) = 0 \end{array}}$ $\Rightarrow \left\{\begin{array}{l} \eta_2(0) = 0 \\ \dot{\eta}_2(0) = 0, \ \dot{\eta}_1(0) = 0 \end{array}\right\} \Rightarrow$

$$\Rightarrow \left\{\begin{array}{l} \eta_2(0) = 0 \Rightarrow c_2^+ + c_2^- = 0 \\ \dot{\eta}_2(0) = 0 \Rightarrow c_2^+ \omega_S - c_2^- \omega_S = 0, \ c_2^+ - c_2^- = 0 \end{array}\right] \Rightarrow \left|\begin{array}{l} c_2^+ = 0 \\ c_2^- = 0 \end{array}\right|$$

$$\Downarrow$$

$$\underbrace{\boxed{\begin{array}{c} \eta_1(t) \text{ oscila} \\ \text{con } \omega_A = \sqrt{\frac{k+2k_{12}}{m}} \end{array}} \Leftarrow \boxed{\begin{array}{c} \eta_2(t) = 0 \\ \forall t \end{array}}}_{\text{Modo normal antisimétrico}}$$

Teoría General de Osciladores Acoplados

→ Sea un <u>sistema conservativo</u> con n grados de libertad, descrito mediante n coordenadas generalizadas q_k $(k = 1, 2, 3, \dots, n)$.

→ Existe una <u>configuración de equilibrio estable</u>:

$$q_{k_0} \;,\; \boxed{\dot{q}_{k_0} = \ddot{q}_{k_0} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \longrightarrow \text{aparecerán } \dot{q}_k \cdot \ddot{q}_k$$

$$\left.\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right|_0 = 0 \Longrightarrow$$

configuración de equilibrio estable

<u>Ec. Lagrange en el equilibrio</u>: $\underbrace{\left(\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right)}$

$$\left.\frac{\partial L}{\partial q_k}\right|_0 = 0$$

$$\Downarrow \quad L = T - V$$

$$\left.\frac{\partial T}{\partial q_k}\right|_0 - \left.\frac{\partial V}{\partial q_k}\right|_0 = 0 \;,\; \boxed{\left.\frac{\partial T}{\partial q_k}\right|_0 = \left.\frac{\partial V}{\partial q_k}\right|_0}$$

→ Ecuaciones de <u>transformación de coordenadas</u> cartesianas a generalizadas <u>independientes del tiempo</u>.

$$\boxed{T = \frac{1}{2}\sum_{j,k} m_{jk}\, \dot{q}_j\, \dot{q}_k}$$

$$\frac{\partial T}{\partial q_k} = 0 \Rightarrow \left.\frac{\partial T}{\partial q_k}\right|_0 = 0 \Longrightarrow \boxed{\left.\frac{\partial V}{\partial q_k}\right|_0 = 0}$$

$$\Downarrow$$

En <u>q_{k_0}</u> el <u>potencial</u> tiene un <u>mínimo</u>

(para demostrarlo habría que ver que $\left.\frac{\partial^2 V}{\partial q_k^2}\right|_0 > 0$)

⊛ Desarrollo de Taylor de V: $\longrightarrow$ en torno al mínimo de potencial

$$V(q_1, q_2, \dots, q_n) =$$

$$= V_0 + \sum_{k=1}^{n} \left.\frac{\partial V}{\partial q_k}\right|_0 (q_k - q_{k_0}) + \frac{1}{2} \sum_{j,k} \left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_0 (q_j - q_{j_0})(q_k - q_{k_0}) + \underbrace{\cancel{\theta(q_k^3)}}_{\text{No consideramos los términos de orden 3 y superior (Oscilaciones pequeñitas)}}$$

Hacemos un cambio de variables $\tilde{q}_k = q_k - q_{k_0}$

y renombramos la nueva variable $\tilde{q}_k$ como q_k.

$$V \simeq V_0 + \frac{1}{2} \sum_{j,k} \underbrace{\left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_0}_{= A_{jk}} q_j q_k$$

$$\boxed{V \simeq V_0 + \frac{1}{2} \sum_{j,k} A_{jk} q_j q_k}$$

⊛ $\boxed{V_0 = V(q_{10}, q_{20}, q_{30}, q_{40}, \dots, q_{n0}) = 0}$ $\longrightarrow$ Elegimos

$$\boxed{L = T - V = \frac{1}{2} \sum_{j,k} m_{jk} \dot{q}_j \dot{q}_k - \frac{1}{2} \sum_{j,k} A_{jk} q_j q_k}$$

Ecuaciones de Lagrange: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{1}{2} \sum_j m_{jk} \dot{q}_j\right) = \frac{1}{2} \sum_j m_{jk} \ddot{q}_j$$

$$\frac{\partial L}{\partial q_k} = -\frac{1}{2} \sum_j A_{jk}\, q_j$$

$$\frac{1}{2} \sum_j m_{jk}\, \ddot{q}_j + \frac{1}{2} \sum_j A_{jk}\, q_j = 0$$

$$\boxed{\sum_j \left(m_{jk}\, \ddot{q}_j + A_{jk}\, q_j \right) = 0} \rightarrow k = 1, \dots, n$$

Probamos una solución de la forma:

$$q_j(t) = a_j\, e^{i(\omega t - \delta)}$$

$$\dot{q}_j(t) = i \omega a_j\, e^{i(\omega t - \delta)}$$

$$\ddot{q}_j(t) = -a_j \omega^2\, e^{i(\omega t - \delta)}$$

$$\sum_j \left[-m_{jk}\, a_j\, \omega^2\, e^{i(\omega t - \delta)} + A_{jk}\, a_j\, e^{i(\omega t - \delta)} \right] = 0$$

$$\underbrace{\boxed{\sum_j \left(A_{jk} - m_{jk}\, \omega^2 \right) a_j = 0} \rightarrow k = 1, \dots, n}_{\text{Sistema de } n \text{ ecuaciones con } n \text{ incógnitas}}$$

Para que tenga solución no trivial:

$$\boxed{\det \left(A_{jk} - m_{jk}\, \omega^2 \right) = 0} \rightarrow n \text{ soluciones} \rightarrow \boxed{\omega_r \quad r = 1, 2, \dots, n}$$

Ecuación característica o secular

Frecuencias características

$$q_j(t) = \sum_{r=1}^{n}\left[a_{jr}\, e^{i(\omega_r t - \delta)} + a_{jr}\, e^{-i(\omega_r t-\delta)}\right]$$

$$\boxed{q_j(t) = \sum_{r=1}^{n} a_{jr} \cos(\omega_r t - \theta_r)} \longrightarrow$$ no son coordenadas normales $\longrightarrow$ a estas las llamaremos η_j

Definimos el vector $\boxed{\vec{q}(t) = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}(t)}$

y las matrices :

$$\{m\}_{n\times n} \equiv (m_{jk})$$
$$\{A\}_{n\times n} \equiv \left(A_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k}\right)$$

Ecuaciones del movimiento : $\{m\}\ddot{\vec{q}} + \{A\}\vec{q} = 0$

Ecuación secular : $\det\left[\{A\} - \omega^2\{m\}\right] = 0$

Soluciones : $\boxed{\vec{q}(t) = \sum_{r=1}^{n} \vec{a}_r \underbrace{\cos(\omega_r t - \delta_r)}_{\eta_r(t)}} \longrightarrow \vec{a}_r = \begin{pmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{nr} \end{pmatrix}$

coordenadas normales $\boxed{\eta_r = \cos(\omega_r t - \delta_r)}$

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

n coordenadas normales $\longleftrightarrow$ n modos normales

ω_r , $r = 1, 2, \ldots, n$

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Sea el $\boxed{\text{modo normal } p}$. $1 \le p \le n \Rightarrow$ Impongámoslo

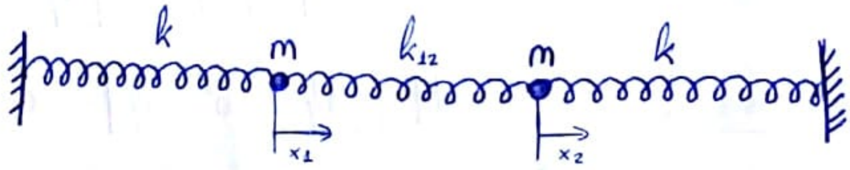
$$\eta_n(t) = 0 \;, \quad \forall n \neq p$$

$$\eta_p(t) \neq 0 \;, \quad \omega_p \longrightarrow \text{frecuencia del modo normal } p.$$

$$\underbrace{\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2p} & \cdots & a_{2n} \\ \cdots & & & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} & \cdots & a_{pn} \\ \cdots & & & & & \\ a_{n1} & a_{n2} & & a_{np} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \eta_p \\ \vdots \\ 0 \end{pmatrix}}$$

$$\left[\begin{array}{ll} q_1 = a_{1p}\eta_p \;\cdots & q_p = a_{pp}\eta_p \\ q_2 = a_{2p}\eta_p \;\cdots & q_n = a_{np}\eta_p \end{array}\right] \quad \frac{q_1}{q_2} = \frac{a_{1p}}{a_{2p}}$$

⊛ Ejemplo:



$$L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 - \frac{1}{2} k_{12} (x_2 - x_1)^2$$

$$\left|\begin{array}{l} q_1 = x_1 \\ q_2 = x_2 \end{array}\right| \Rightarrow \text{Matriz } \{m\} \longrightarrow \{m\} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$T = \frac{1}{2} \sum_{j,k} m_{jk} \dot{q}_j \dot{q}_k$$

$$T = \frac{1}{2} m_{11} \dot{x}_1^2 + \frac{1}{2} m_{12} \dot{x}_1 \dot{x}_2 + \frac{1}{2} m_{21} \dot{x}_1 \dot{x}_2 + \frac{1}{2} m_{22} \dot{x}_2^2 =$$

$$= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$$

$$\text{Matriz } \{A\} \longrightarrow \{A\} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k} \qquad A_{11} = \frac{\partial^2 V}{\partial x_1^2} \qquad A_{22} = \frac{\partial^2 V}{\partial x_2^2} \qquad A_{12} = A_{21} = \frac{\partial^2 V}{\partial x_1 \partial x_2}$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k_{12} (x_2 - x_1)^2$$

$$\frac{\partial V}{\partial x_1} = k x_1 - k_{12}(x_2 - x_1)$$

$$\frac{\partial V}{\partial x_2} = k x_2 + k_{12}(x_2 - x_1)$$

$$\begin{cases} A_{11} = \dfrac{\partial^2 V}{\partial x_1^2} = k + k_{12} \\ A_{21} = A_{12} = -k_{12} = \dfrac{\partial^2 V}{\partial x_1 \partial x_2} \\ A_{22} = \dfrac{\partial^2 V}{\partial x_2^2} = k + k_{12} \end{cases}$$

$$\{A\} = \begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}$$

Ecuación secular: $\boxed{\det\left[\{A\} - \omega^2 \{m\}\right] = 0}$

→ Resolvemos para obtener las frecuencias características.

$$\begin{vmatrix} k + k_{12} - \omega^2 m & -k_{12} \\ -k_{12} & k + k_{12} - \omega^2 m \end{vmatrix} = 0$$

$$\boxed{(k + k_{12} - \omega^2 m)^2 - k_{12}^2 = 0}$$ Ecuación secular

$$(k + k_{12} - \omega^2 m)^2 = k_{12}^2$$

$$k + k_{12} - \omega^2 m = \pm k_{12}$$

$\oplus$ $k + k_{12} - \omega_1^2 m = +k_{12}$, $\quad k = \omega_1^2 m$, $\quad \boxed{\omega_1 = \sqrt{\frac{k}{m}} = \omega_S}$

$\ominus$ $k + k_{12} - \omega_2^2 m = -k_{12}$, $\quad k + 2k_{12} = \omega_2^2 m$, $\quad \boxed{\omega_2 = \sqrt{\frac{k + 2k_{12}}{m}} = \omega_A}$

Coordenadas normales:

$$\eta_1(t) = A_S \cos(\omega_S t - \delta_S) \qquad \eta_2(t) = A_A \cos(\omega_A t - \delta_A)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \longrightarrow \underbrace{\left\{ \begin{array}{l} x_1(t) = a_{11}\,\eta_1(t) + a_{12}\,\eta_2(t) \\ x_2(t) = a_{21}\,\eta_1(t) + a_{22}\,\eta_2(t) \end{array} \right.}_{\text{Cualquier solución del sistema es combinación lineal de las coordenadas normales.}}$$

$$\vec{a}_n = \left\{ \begin{array}{l} \vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \\ \vec{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \end{array} \right.$$

Modos normales:

o $n = 1 \longrightarrow \eta_2(t) = 0 \quad \forall t \quad , \quad \eta_1(t) \neq 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} \Rightarrow \underbrace{\left| \begin{array}{l} x_1 = a_{11}\,\eta_1 \\ x_2 = a_{21}\,\eta_1 \end{array} \right|}$$

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \qquad \boxed{\frac{x_1}{x_2} = \frac{a_{11}}{a_{21}}}$$

$\downarrow$

verifica: $\quad \left[\{A\} - \omega_1^2 \{m\} \right] \vec{a}_1 = 0 \qquad \omega_1^2 = \frac{k}{m}$

$$\begin{pmatrix} k + k_{12} - \frac{k}{m} m & -k_{12} \\ -k_{12} & k + k_{12} - \frac{k}{m} m \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$$

$$\begin{pmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$$

$$k_{12}(a_{11} - a_{21}) = 0 \xRightarrow{(k_{12} \neq 0)} \boxed{a_{11} = a_{21}} = a$$

$$\boxed{\frac{x_2}{x_1} = \frac{a}{a} = 1} \iff \boxed{x_1 = x_2 \quad \omega_1 = \sqrt{\frac{k}{m}}}$$

o $n = 2 \longrightarrow \eta_1(t) = 0 \quad \forall t \; , \; \eta_2(t) \neq 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} \Rightarrow \underbrace{\left\{ \begin{array}{l} x_1 = a_{12}\eta_2 \\ x_2 = a_{22}\eta_2 \end{array} \right\}}_{\boxed{\frac{x_1}{x_2} = \frac{a_{12}}{a_{22}}}}$$

$$\vec{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$\downarrow$

verifica: $\left[\{A\} - \omega_2^2 \{m\} \right] \vec{a}_2 = 0 \qquad \omega_2^2 = \frac{k + 2k_{12}}{m}$

$$\begin{pmatrix} k + k_{12} - \frac{k+2k_{12}}{m} m & -k_{12} \\ -k_{12} & k + k_{12} - \frac{k+2k_{12}}{m} m \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0$$

$$\begin{pmatrix} -k_{12} & -k_{12} \\ -k_{12} & -k_{12} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0$$

$$-k_{12}(a_{12} + a_{22}) = 0 \xRightarrow{(k_{12} \neq 0)} a_{12} + a_{22} = 0$$

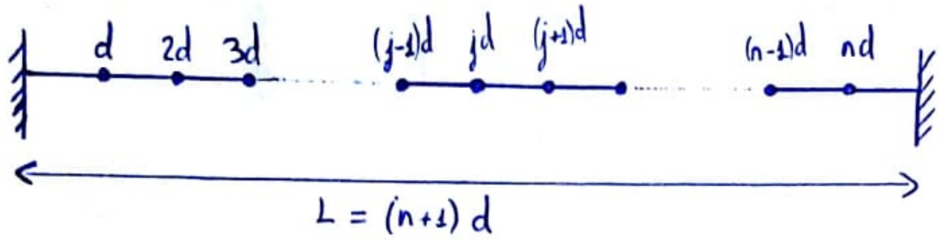
$$\boxed{a_{12} = -a_{22} = a'}$$

$$\boxed{\frac{x_1}{x_2} = \frac{a_{12}}{a_{22}} = \frac{a'}{-a'} = -1} \iff \boxed{x_1 = -x_2 \quad \omega_A = \sqrt{\frac{k+2k_{12}}{m}}}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & a' \\ a & -a' \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}} \rightarrow (x_1, x_2) \longrightarrow (\eta_1, \eta_2)$$

La cuerda discreta

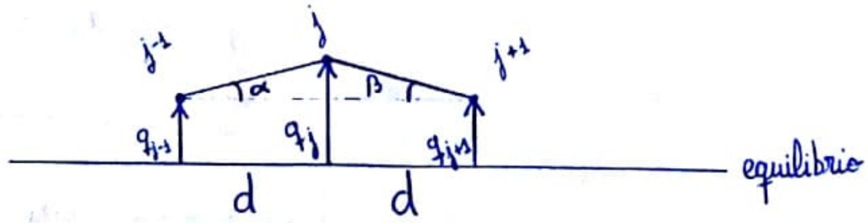
Sean $\underline{n}$ partículas de masa $\underline{m}$, equiespaciadas, separadas una distancia $\underline{d}$.



Aproximaciones
- Solo hay desplazamientos verticales.
- Solo hay interacción entre los vecinos más próximos
  - ↳ la fuerza sobre la masa j solo depende de las posiciones de la $j-1$ y la $j+1$.
- Desplazamientos pequeños

n grados de libertad ⟶ n coordenadas generalizadas:

$$q_j \quad . \quad j = 1, \dots, n$$



→ Ecuación de Newton:

$\tau \equiv$ tensión de la cuerda

Para desplazamientos pequeños consideramos τ cte e igual a la tensión en equilibrio.

La fuerza sobre la masa j (F_j) es la debida a las masas $j-1$ y $j+1$.

$$F_j = -\tau \sin\alpha - \tau \sin\beta$$

Desplazamientos pequeños $\Rightarrow$ $\alpha, \beta \ll 1 \Rightarrow \begin{cases} \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} \simeq \sin\alpha \\ \tan\beta = \dfrac{\sin\beta}{\cos\beta} \simeq \sin\beta \end{cases}$

$$F_j \simeq -\tau \tan\alpha - \tau \tan\beta \qquad \begin{cases} \tan\alpha = \dfrac{q_j - q_{j-1}}{d} \\ \tan\beta = \dfrac{q_j - q_{j+1}}{d} \end{cases}$$

$$F_j \simeq -\tau\left(\frac{q_j - q_{j-1}}{d}\right) - \tau\left(\frac{q_j - q_{j+1}}{d}\right)$$

$$\boxed{F_j = m\ddot{q}_j = \frac{\tau}{d}\left(q_{j-1} - 2q_j + q_{j+1}\right)}$$

Ecuación del movimiento ($j = 1, 2, \dots, n$)

Probamos la solución:

$$q_j(t) = a_j e^{i\omega t} \qquad a_j \in \mathbb{C}$$

$$\dot{q}_j(t) = i\omega a_j e^{i\omega t} \qquad \ddot{q}_j(t) = -\omega^2 a_j e^{i\omega t}$$

$$-m\omega^2 a_j e^{i\omega t} = \frac{\tau}{d}\left(a_{j-1} e^{i\omega t} - 2a_j e^{i\omega t} + a_{j+1} e^{i\omega t}\right)$$

$$-m\omega^2 a_j = \frac{\tau}{d}\left(a_{j-1} - 2a_j + a_{j+1}\right)$$

Sistema de n ecuaciones con n incógnitas

$$(1) \quad \boxed{-\frac{\tau}{d} a_{j-1} + \left(\frac{2\tau}{d} - m\omega^2\right) a_j - \frac{\tau}{d} a_{j+1} = 0} \quad \boxed{j = 1, 2, \dots, n}$$

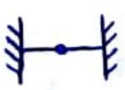
Definimos $\lambda \equiv \dfrac{2\tau}{d} - m\omega^2$

$$-\frac{\tau}{d}\, a_{j-1} + \lambda\, a_j - \frac{\tau}{d}\, a_{j+1} = 0 \qquad (j = 1, 2, \ldots, n)$$

Para que el sistema tenga solución no trivial $\longrightarrow$ S.C.I.

$$\det \begin{pmatrix} \lambda & -\frac{\tau}{d} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ -\frac{\tau}{d} & \lambda & -\frac{\tau}{d} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & -\frac{\tau}{d} & \lambda & -\frac{\tau}{d} & \cdots & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & -\frac{\tau}{d} & \lambda \end{pmatrix} = 0$$

$\longrightarrow$ soluciones ω_n (frecuencias propias del sistema):

- $\boxed{n = 1}$  $\det(\lambda) = 0 \Longrightarrow \lambda = 0 = \dfrac{2\tau}{d} - m\omega_1^2$

$$\frac{2\tau}{d} = m\omega_1^2$$

$$\boxed{\omega_1 = \sqrt{\frac{2\tau}{md}}}$$

- $\boxed{n = 2}$  $\det\begin{pmatrix} \lambda & -\frac{\tau}{d} \\ -\frac{\tau}{d} & \lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - \dfrac{\tau^2}{d^2} = 0$

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$$\left(\frac{2Z}{d} - m\,\omega_{1,2}^2\right)^2 - \frac{Z^2}{d^2} = 0$$

$$\left(\frac{2Z}{d} - m\,\omega_{1,2}^2\right)^2 = \frac{Z^2}{d^2}, \quad \frac{2Z}{d} - m\,\omega_{1,2}^2 = \pm\frac{Z}{d}$$

$\boxed{\oplus}$ $\dfrac{2Z}{d} - m\omega_1^2 = \dfrac{Z}{d}$

$$\frac{Z}{d} = m\omega_1^2$$

$$\omega_1 = \sqrt{\frac{Z}{dm}} \underset{\left(k=\frac{Z}{d}\right)}{=} \sqrt{\frac{k}{m}}$$

$\boxed{\ominus}$ $\dfrac{2Z}{d} - m\omega_2^2 = -\dfrac{Z}{d}$

$$\frac{3Z}{d} = m\omega_2^2$$

$$\omega_2 = \sqrt{\frac{3Z}{dm}} \underset{\left(k=\frac{Z}{d}\right)}{=} \sqrt{\frac{3k}{m}}$$

o Para $\boxed{n \geq 3}$:

$\boxed{\text{Marion}} \longrightarrow \boxed{a_j = a\, e^{i(j\gamma - \delta)}}$ $\quad a \in \mathbb{R}, \ \gamma \in \mathbb{R}, \ \delta \in \mathbb{R}$

Sustituyendo este a_j en la ecuación (1):

$$-\frac{Z}{d}\, a\, e^{i((j-1)\gamma - \delta)} + \left(\frac{2Z}{d} - m\omega^2\right) a\, e^{i(j\gamma - \delta)} - \frac{Z}{d} a e^{i((j+1)\gamma - \delta)} = 0$$

$$-\frac{Z}{d}\, a\, e^{i(j\gamma - \delta)} \cdot e^{-i\gamma} + \left(\frac{2Z}{d} - m\omega^2\right) a\, e^{i(j\gamma - \delta)} - \frac{Z}{d} a\, e^{i(j\gamma - \delta)} e^{i\gamma} = 0$$

$$-\frac{Z}{d}\, a \left[e^{-i\gamma} + e^{i\gamma}\right] + \left(\frac{2Z}{d} - m\omega^2\right) a = 0$$

$$-\frac{z}{d}\left[e^{i\gamma}+e^{-i\gamma}\right]+\frac{2z}{d}-m\omega^2=0$$

$\circledast^1$ $\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$

$$-\frac{z}{d}\,2\cos\gamma+\frac{2z}{d}-m\omega^2=0$$

$\circledast^2$ $\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}$

$$m\omega^2=\frac{2z}{d}\left(1-\cos\gamma\right)$$

$$m\omega^2=\frac{2z}{d}\,2\sin^2\frac{\gamma}{2}$$

$$\boxed{\omega^2=\frac{4z}{md}\sin^2\frac{\gamma}{2}}$$

En cada caso (con n g.l.) hay n soluciones:

$$\boxed{\omega_r = 2\sqrt{\frac{z}{md}}\,\sin\frac{\gamma_r}{2}} \qquad r=1,2,\dots,n$$

Frecuencias características propias

Marion $\longrightarrow$ $\boxed{a_{jr} = a_r\, e^{i(j\gamma_r-\delta_r)}}$

$\gamma_r, \delta_r \longrightarrow$ condiciones de contorno (extremos de la cuerda fijos)

$a_r \longrightarrow$ condiciones iniciales $q_j(t=0)$, $\dot{q}_j(t=0)$

? $\delta_r = \left(n+\frac{1}{2}\right)\frac{\pi}{2}$ $\qquad \gamma_r = \frac{r\pi}{n+1}$

$$q_j(t) = \sum_{n=1}^{n} a_{jn}\, e^{i\omega_n t} = \sum_{n=1}^{n} a_n \sin\left(j\,\frac{n\pi}{n+1}\right) e^{i\omega_n t}$$

nº de la masa

nos quedamos con la parte real

$\sin(\gamma_n j)$

⊛ $a_{jn} = a_n\, e^{i(\gamma_n j - \delta_n)}$

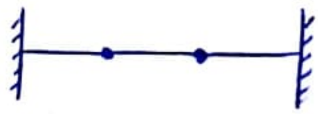
$\hookrightarrow \mathrm{Re}\left[e^{i(\gamma_n j - \delta_n)}\right] = \cos\left(\gamma_n j - \frac{\pi}{2}\right)$

n

↳ nº de grados de libertad, nº de frecuecias propias.

$$\boxed{\begin{gathered} q_j(t) = \sum_{n=1}^{n} a_n \sin\left(j\,\frac{n\pi}{n+1}\right)\cos(\omega_n t - \theta_n) \\ a_n,\ \theta_n \in \mathbb{R} \qquad \omega_n = 2\,\omega_0 \sin\left(\frac{n\pi}{2(n+1)}\right) \\ \omega_0 = \sqrt{\frac{Z}{dm}} \qquad n = 1, 2, \ldots, n \end{gathered}}$$

⊛ Ejemplo $\boxed{n=2}$



q_1, q_2

m_1, m_2

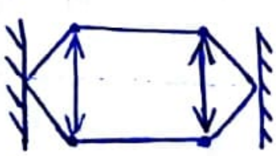
$\omega_1, \omega_2 \longrightarrow$ 2 modos normales.

Modo $\boxed{n=1}$

$$\omega_1 = 2\,\omega_0 \sin\frac{\pi}{2(2+1)} = 2\,\omega_0 \sin\frac{\pi}{6} = \omega_0 = \sqrt{\frac{Z}{md}}$$

$$q_1(t) = a_1 \sin\frac{\pi}{3}\cos(\omega_1 t - \theta_1) = a_1 \frac{\sqrt{3}}{2}\cos(\omega_1 t - \theta_1)$$

$$q_2(t) = a_1 \sin\frac{2\pi}{3}\cos(\omega_1 t - \theta_1) = a_1 \frac{\sqrt{3}}{2}\cos(\omega_1 t - \theta_1)$$



Modo $\boxed{n=2}$ $\quad \omega_2 = 2\omega_0 \sin\frac{2\pi}{2(2+1)} = 2\omega_0 \sin\frac{\pi}{3} = \sqrt{3}\,\omega_0 =$

$$= \sqrt{\frac{3\,\mathcal{Z}}{md}}$$

$$q_1(t) = a_2 \sin\frac{2\pi}{3}\cos(\omega_2 t-\theta_2) = a_2\frac{\sqrt{3}}{2}\cos(\omega_2 t-\theta_2)$$

$$q_2(t) = a_2 \sin\frac{4\pi}{3}\cos(\omega_2 t-\theta_2) = -a_2\frac{\sqrt{3}}{2}\cos(\omega_2 t-\theta_2)$$

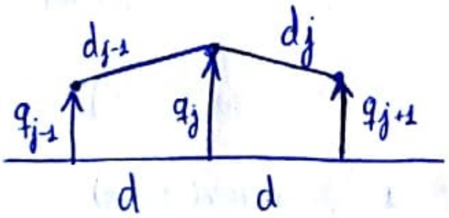
Solución general:

Combinación lineal de los modos normales:

$$q_1(t) = \frac{\sqrt{3}}{2}\left[a_1\cos(\omega_1 t-\theta_1) + a_2\cos(\omega_2 t-\theta_2)\right]$$

$$q_2(t) = \frac{\sqrt{3}}{2}\left[a_1\cos(\omega_1 t-\theta_1) - a_2\cos(\omega_2 t-\theta_2)\right]$$

→ Resolución por Lagrange del problema de la cuerda discreta:



$$d_j^2 = d^2 + (q_j - q_{j+1})^2$$

$$d_j = \sqrt{d^2 + (q_j - q_{j+1})^2}$$

$$d_{j-1} = \sqrt{d^2 + (q_j - q_{j-1})^2}$$

Consideramos cada tramo de cuerda como un muelle de $\boxed{l_0 = a}$

$$T = \frac{1}{2} \sum_{j=1}^{n+1} m \dot{q}_j^2$$

⊛ $\underbrace{q_0 = q_{n+1} = \cdots = 0}_{\text{no varían el sistema}}$

$$V = \frac{1}{2} \sum_{j=1}^{n+1} k (d_j - a)^2 = \frac{1}{2} \sum_{j=1}^{n+1} k \left[\sqrt{d^2 + (q_{j+1} - q_j)^2} - a\right]^2$$

$$L = T - V = \frac{1}{2} \sum_{j=1}^{n+1} \left[m \dot{q}_j^2 - k \left(\sqrt{d^2 + (q_{j+1} - q_j)^2} - a \right)^2 \right]$$

Taylor:

$$d_j = \sqrt{d^2 + \underbrace{(q_{j+1} - q_j)^2}_{\Delta}} = \sqrt{d^2 + \Delta^2} =$$

$\boxed{\text{⊛ Desarrollo de Taylor en torno a } \Delta = 0 \text{ (pequeñas oscilaciones)}}$

(orden 2)

$$\simeq d + \frac{1}{2} \frac{\Delta^2}{d}$$

$$d_j = \sqrt{d^2 + \Delta^2} \quad , \quad d_j(\Delta = 0) = d$$

$$\left.\frac{d(d_j)}{d\Delta}\right|_{\Delta=0} = \left.\frac{\Delta}{\sqrt{d^2 + \Delta^2}}\right|_{\Delta=0} = 0$$

$$\left.\frac{d^2(d_j)}{d\Delta^2}\right|_{\Delta=0} = \left.\frac{\sqrt{d^2 + \Delta^2} - \Delta \frac{\Delta}{\sqrt{d^2+\Delta^2}}}{d^2 + \Delta^2}\right|_{\Delta=0} = \frac{d}{d^2} = \frac{1}{d}$$

$$V \simeq \frac{1}{2} k \sum_{j=1}^{n+1} \left[(d-a) + \frac{1}{2} \frac{\Delta^2}{d} \right]^2 = \frac{1}{2} k \sum_{j=1}^{n+1} \left[(d-a)^2 + \underbrace{\frac{1}{4} \frac{\Delta^4}{d^2}}_{\to 0 \text{ (despreciamos)}} + (d-a) \frac{\Delta^2}{d} \right] =$$

$$= \frac{1}{2} k \sum_{j=1}^{n+1} \left[(d-a)^2 + \frac{(d-a)}{d} \Delta^2 \right] = \text{cte} + \frac{1}{2} \frac{d-a}{d} k \sum_{j=1}^{n+1} \Delta^2$$

$$V' = \frac{1}{2} \frac{d-a}{d} k \sum_{j=1}^{n+1} (q_{j+1} - q_j)^2$$

$$T = \frac{1}{2} m \sum_{j=1}^{n+1} \dot{q}_j^2$$

$$L = T - V =$$

$$= \frac{1}{2}\sum_{j=1}^{n+1}\left[m\dot{q}_j^2 - \underbrace{\frac{d-a}{d}k}_{\hat{k}}(q_{j+1}-q_j)^2\right] =$$

$$= \frac{1}{2}\sum_{j=1}^{n+1}\left[m\dot{q}_j^2 - \hat{k}\,(q_{j+1}-q_j)^2\right]$$

$$\frac{\partial L}{\partial q_j} = -\frac{1}{2}\hat{k}\,2\,(q_{j+1}-q_j)\,(-1) + \frac{1}{2}2\hat{k}\,(q_j - q_{j-1})\,(-1) =$$

$$= \hat{k}(q_{j+1} - 2q_j + q_{j-1})$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{1}{2}\,2m\,\dot{q}_j = m\dot{q}_j$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = m\ddot{q}_j$$

Ec. Lagrange q_j : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

$$m\ddot{q}_j = \hat{k}\,(q_{j+1} - 2q_j + q_{j-1})$$

$$\ddot{q}_j = \frac{\hat{k}}{m}(q_{j+1} - 2q_j + q_{j-1}) \qquad \omega_0 = \sqrt{\frac{\hat{k}}{m}}$$

$$\boxed{\ddot{q}_j = \omega_0^2\,(q_{j+1} - 2q_j + q_{j-1})} \qquad \hat{k} = \frac{Z}{d}$$

$$\ddot{q}_j = \frac{Z}{md}(q_{j+1} - 2q_j + q_{j-1})$$

$\longrightarrow$ A partir de aquí es igual que antes

3. Oscilacións lineais

3.1. Probar que as seguintes expresións para a solución dun oscilador harmónico nunha dimensión son equivalentes. Expresar as constantes de cada unha delas en función das da anterior.

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} = B_1 \cos \omega t + B_2 \operatorname{sen} \omega t =$$

$$= A \cos(\omega t - \delta) = \mathrm{Re}\,[C e^{i\omega t}]$$

3.2. A enerxía potencial dunha masa m nunha dimensión, a distancia r da orixe, é:

$$U(r) = U_0(\frac{r}{R} + \lambda^2 \frac{R}{r})$$

con U_0, R , λ constantes positivas. Atopar a posición de equilibrio r_0. Amosar que, para pequenas desviacións do equilibrio, o potencial é harmónico e calcular a frecuencia das pequenas oscilacións.

3.3. Un arco exerce unha forza de 134 N sobre unha frecha que ten a base situada a 0.72 m da posición de equilibrio do arco. Tendo en conta que a masa da frecha é de 23 g, *a)* calcular a velocidade da frecha cando sae do arco; *b)* calcular o tempo no que a frecha é propulsada.

3.4. Escribir as ecuacións do movemento para os sistemas da figura:

3.5. Un resorte ideal de masa desprezable, constante k e lonxitude natural l_0, unido nos seus extremos a dúas masas m e M, atópase suspendido do teito polo extremo da masa m. En $t = 0$ sóltase o resorte. Analisar o movemento resultante.

3.6. Calcular a densidade de probabilidade de que un oscilador harmónico se atope na posición x.

3.7. O peche automático dunha porta pode ser considerado como un oscilador amortecido (un resorte cun émbolo que proporciona un rozamento proporcional á velocidade). Supoñendo que o sistema da porta co peche é equivalente a unha masa de 10 kg en contacto cun resorte de constante $k = 10$ N cm^{-1}, a) calcular o coeficiente de rozamento b (definido por $F_r = -bv$) necesario para obter amortecemento crítico.

Con este resorte e un émbolo de $b = 2000\ kg\ s^{-1}$ diséñase un sistema de xeito que coa porta pechada o resorte está sufrindo unha elongación de 0.5 cm. e coa porta aberta de 10 cm. b) Calcular a velocidade final do sistema equivalente ao pechar a porta. ¿Canto tarda en pechar?. c) ¿Qué coeficiente de rozamento é necesario para que tarde o dobre?. ¿Qué tipo de oscilador sería?. d) Se cambiamos a porta por unha que pese a metade, ¿cómo varían a velocidade e o tempo de peche?.

3.8. Unha masa de 1000 kg cae dende unha altura de 10 m sobre unha plataforma de masa desprezable montada sobre un resorte de constante elástica k cun sistema de amortecemento de constante b. O sistema deséñase de tal xeito que a plataforma alcance a nova posición de equilibrio 0.2 m por debaixo da súa posición orixinal, o máis rápidamente posible despóis do impacto e sen oscilar.

a) Obter a constante do resorte k e a constante de amortecemento b.

b) Escribir a solución $x(t)$ a partires do momento do impacto, asegurándose de calcular as constantes para que se satisfagan as condicións iniciais correctas.

3.9. Un oscilador libre ten periodo $\tau_0 = 1.000\, s$. Engádese un pequeno amortecemento e o seu periodo cambia a $\tau_1 = 1.001\, s$ Obter o factor de amortecemento. ¿En qué factor mingua a amplitude despóis de 10 ciclos?

3.10. Un oscilador harmónico con amortecemento sométese a unha forza externa $F(t) = fe^{-\mu t}$. Analisar o seu movemento supoñendo que $\gamma^2 > \omega_0^2$.

3.11. Sabemos que para un oscilador forzado, a máxima resposta (A^2) ocorre a $\omega \simeq \omega_0$ para $\gamma \ll \omega_0$. Probar que A^2 é igual á metade do seu valor máximo cando $\omega \simeq \omega_0 \pm \gamma$, de xeito que a anchura é 2γ

3.12. *a)* Atopar as condicións inciais dun oscilador harmónico infraamortecido de tal xeito que alcance inmediatamente a situación estacionaria cando é perturbado (forzado) cunha forza $F(t) = F_0 \cos \omega t$.

b) Calcular o transitorio se en $t = 0$ estaba en repouso.

c) Calcular a potencia media transferida ao oscilador por esta forza.

d) ¿Cando se fai máxima a potencia?

e) Demostrar que a potencia media disipada polo rozamento é a mesma que a producida pola forza F.

3.13. Calcular os coeficientes de Fourier das seguintes funcións periódicas:
a) $f(t) = \sin^2(t)$; b) $f(t) = \sin^3(t)$; c) $f(t) = t, \quad -1 < t < 1$;
d)

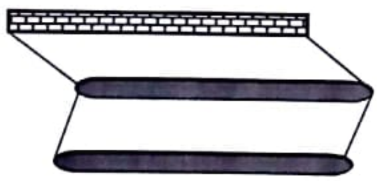
$$f(t) = \begin{cases} 1+\mathrm{t}, & \text{se } -1 \leq t \leq 0; \\ 1-\mathrm{t}. & \text{se } 0 \leq t \leq 1. \end{cases}$$

e)

$$f(t) = \begin{cases} \text{sen } \pi \mathrm{t}, & \text{se } 0 \leq t \leq 1; \\ 0, & \text{se } 1 \leq t \leq 2. \end{cases}$$

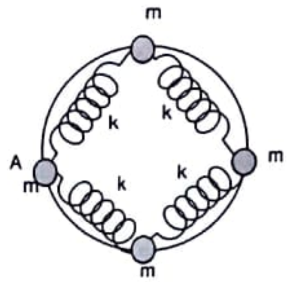
√**3.14.** Unha variña homoxénea de masa M e lonxitude L está suspendida dun dos seus extremos dun arame parabólico de ecuación $y = ax^2$ con $a > 0$. O extremo da variña pode esvarar sen rozamento polo arame. Calcúlense as frecuencias propias das oscilacións pequenas ao redor da posición de equilibrio. O momento de inercia dunha vara homoxénea respecto o seu centro de masas é $I = 1/12ML^2$.

√ **3.15.** Unha barra homoxénea de masa m_1 pendúrase do teito mediante dous fíos unidos ós seus extremos. Os fíos teñen lonxitude a e peso desprezable. Outra barra homoxénea da mesma lonxitude e masa m_2 suspéndese da primeira tamén mediante fíos de lonxitude a e peso desprezable. O sistema oscila no plano vertical coas barras sempre horizontais. Obténanse as frecuencias das pequenas oscilacións do sistema.



√**3.16.** Estúdiense as vibracións lonxitudinais dunha molécula triatómica considerada como un átomo central de masa M ligado mediante resortes de constante k a dous átomos laterais de masa m.

√3.17. Tres masas iguais están conectadas mediante resortes e móvense nun círculo, como se amosa na figura. O punto A está fixo. Atópense as posicións de equilibrio estable, as coordenadas normais e as frecuencias propias do sistema.



√3.18. Considérese unha corda discreta composta de tres partículas. No instante inicial, a partícula central sofre un desprazamento a e déixase libre dende o repouso. Calcúlese o movemento posterior do sistema.

√3.19 Un aro delgado de radio R e masa M pode oscilar no seu propio plano, estando fixo un dos seus puntos. Suxeita ao aro está unha masa M limitada a moverse (sen rozamento) sobre o mesmo. Supoñendo que só pode haber oscilacións de pequena amplitude, obter as pulsacións propias do sistema e determinar os conxuntos de condicións iniciais que permiten ao sistema oscilar cos seus modos normais. O momento de inercia do aro con respecto a un punto do propio aro é $I = 2MR^2$

3.20. Sexa $\phi(x,t)$ a variable dinámica dun sistema continuo con densidade lagrangiana

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \cos\phi.$$

Obténanse as ecuacións de Lagrange do sistema.

3.21. Demostrar que se se toman ψ, ψ^* como dúas variables de campo independientes, a densidade lagrangiana

$$\mathcal{L} = \frac{\hbar^2}{2m}\nabla\psi\cdot\nabla\psi^* + V\psi\psi^* + \frac{\hbar}{2i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

conduce á ecuación de Schrodinger.

3Ejs

Boletín oscilaciones lineales

① Probar la equivalencia entre las siguientes soluciones del oscilador armónico unidimensional ($m\ddot{x} + kx = 0$)

Ⓘ $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

Fórmula de Euler : $\boxed{e^{i\alpha} = \cos\alpha + i\sin\alpha}$

$$x(t) = C_1(\cos\omega t + i\sin\omega t) + C_2(\cos(-\omega t) + i\sin(-\omega t)) =$$

$$= \underbrace{(C_1 + C_2)}_{B_1}\cos(\omega t) + \underbrace{(C_1 - C_2)i}_{B_2}\sin(\omega t) =$$

$$= B_1\cos\omega t + B_2\sin\omega t \quad ✓$$

Ⓘⓘ $x(t) = B_1\cos\omega t + B_2\sin\omega t$

con $B_1 = C_1 + C_2$, $B_2 = (C_1 - C_2)i$

$$x(t) = B_1\cos\omega t + B_2\sin\omega t = A\left(\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right) =$$

(*) $\left.\begin{array}{l}\left(\frac{B_1}{A}\right)^2 + \left(\frac{B_2}{A}\right)^2 = 1 \\ \boxed{\sqrt{B_1^2 + B_2^2} = A}\end{array}\right\} \Rightarrow \left.\begin{array}{l}\frac{B_1}{A} = \cos\delta \\ \frac{B_2}{A} = \sin\delta\end{array}\right\} \Rightarrow \tan\delta = \frac{B_2}{B_1}$, $\delta = \arctan\frac{B_2}{B_1}$

$$= A(\cos\delta\cos\omega t + \sin\delta\sin\omega t) = A\cos(\omega t - \delta) \quad ✓$$

Ⓘⓘⓘ $x(t) = A\cos(\omega t - \delta)$ con $A = \sqrt{B_1^2 + B_2^2}$ y $\delta = \arctan\frac{B_2}{B_1}$

$$x(t) = A\cos(\omega t - \delta) = R_e\left[A e^{i(\omega t-\delta)}\right] = R_e\left[A e^{-i\delta} e^{i\omega t}\right] =$$

$$= R_e\left[C e^{i\omega t}\right] \quad ✓ \qquad \boxed{C = A e^{-i\delta}}$$

$$|C| = A = \sqrt{R_e^2(C) + I_m^2(C)}$$

$$\delta = \arccos\left(\frac{R_e(C)}{|C|}\right) = \arcsin\left(-\frac{I_m(C)}{|A|}\right)$$

IV $\quad x(t) = R_e\left[C e^{i\omega t}\right]$

con $\quad C = A e^{-i\delta} \quad \left| \begin{array}{l} A = \sqrt{B_1^2 + B_2^2} \\ \delta = \arctan \frac{B_2}{B_1} \end{array}\right.$

3.2 Energía potencial de una masa m a una distancia r del origen:

$$U(r) = U_0\left(\frac{r}{R} + \lambda^2 \frac{R}{r}\right) \qquad U_0, R, \lambda > 0$$

¿Posición de equilibrio r_0? $\longrightarrow$ Mínimo de $U(r)$

Ver que el potencial es armónico para pequeñas desviaciones del equilibrio $\longrightarrow$ ¿frecuencia de las pequeñas oscilaciones?

Buscamos un mínimo del potencial:

$$\frac{dU}{dr} = U_0\left(\frac{1}{R} + \lambda^2 R\left(-\frac{1}{r^2}\right)\right) = U_0\left(\frac{1}{R} - \frac{\lambda^2 R}{r^2}\right)$$

$$\left.\frac{dU}{dr}\right|_{r_0} = 0 = U_0\left(\frac{1}{R} - \frac{\lambda^2 R}{r_0^2}\right) \qquad \frac{U_0}{R} = \frac{U_0 \lambda^2 R}{r_0^2}$$

$$r_0^2 = \lambda^2 R^2$$

$$\boxed{r_0 = \pm \lambda R}$$

$(r > 0)$

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$$\frac{d^2U}{dr^2} = U_0 \lambda^2 R \frac{2r}{r^4} = 2U_0\lambda^2 R \frac{1}{r^3}$$

$$\left.\frac{d^2U}{dr^2}\right|_{r_0=\lambda R} > 0 \qquad \cancel{\left.\frac{d^2U}{dr^2}\right|_{r_0=-\lambda R} < 0}$$

además
r tiene que ser positivo por definición

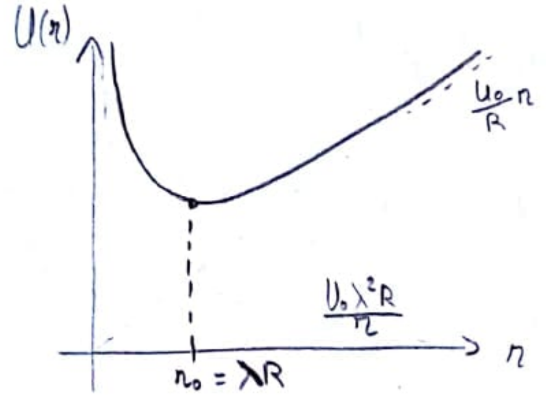
$\Downarrow$

mínimo

$\Downarrow$

$$\boxed{r_0 = \lambda R}$$

$$\left.\frac{d^2U}{dr^2}\right|_{r_0} = 2U_0\lambda^2 R \frac{1}{\lambda^3 R^3} = \frac{2U_0}{\lambda R^2}$$



Desarrollo de Taylor de $U(r)$ en torno a r_0:

$$U(r) \simeq U(r_0) + \left.\frac{dU}{dr}\right|_{r_0}^{\,0}(r-r_0) + \frac{1}{2}\left.\frac{d^2U}{dr^2}\right|_{r_0}(r-r_0)^2 + \underbrace{\cancel{\mathcal{O}(r^3)}}_{}$$

Pequeñas oscilaciones, no consideramos términos de orden superior

$$U(r_0) = U_0\left(\frac{r_0}{R} + \lambda^2\frac{R}{r_0}\right) =$$

$$= U_0\left(\frac{\lambda R}{R} + \lambda^2\frac{R}{\lambda R}\right) = U_0(\lambda + \lambda) = 2\lambda U_0$$

$$\boxed{U(r)} \simeq \underbrace{2\lambda U_0}_{cte} + \frac{U_0}{\lambda R^2}(r-r_0)^2 = \boxed{\frac{1}{2}k(r-r_0)^2 + cte}$$

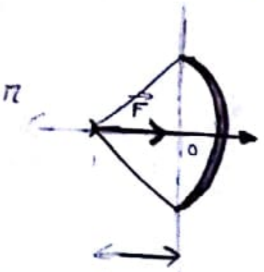
$\hookrightarrow$ potencial armónico $\quad \boxed{k = \frac{2U_0}{\lambda R^2}}$

$$\boxed{\omega = \sqrt{\frac{2U_0}{\lambda R^2 m}}} \longleftarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

(3.3) Un arco ejerce $\boxed{F = 134\ N}$ sobre una flecha cuya base está a $\boxed{0'72\ m}$ de la posición de equilibrio del arco

$$\boxed{m_{flecha} = 23g}$$

a) ¿ velocidad de la flecha al salir del arco?



$r_i = 0'72m$

$r_0 = 0$

$V_0 = 0$

$$\vec{F}_i = -134N\,\hat{r}$$

$r \equiv$ distancia a la posición de equilibrio

$$F(r) = -\frac{dV}{dr}$$

$$V(r) = V_0 + \frac{1}{2}k(r - r_0)^2 = \frac{1}{2}kr^2 \longrightarrow \text{Energía potencial}$$

$$F(r) = -kr$$

$$k = -\frac{F(r)}{r} = -\frac{F_i}{r_i} = \frac{134N}{0'72m} = 186'\hat{1}\ \frac{N}{m}$$

$$T(r) = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}mv^2$$

Conservación de la energía:

$$\cancelto{0}{T(r=0'72m)} + V(r=0'72m) = T(r=0m) + \cancelto{0}{V(r=0m)}$$

$$\frac{1}{2}kr^2 = \frac{1}{2}mv^2$$

$$kr^2 = mv^2 \quad , \quad v = \sqrt{\frac{kr^2}{m}}$$

$$v = \sqrt{\frac{186'\hat{1}\ N/m \cdot (0'72m)^2}{0'023\ Kg}} \simeq 64'77\ m/s$$

$$\boxed{V = 64'77\ m/s\ \frac{1\ Km}{10^3 m} \cdot \frac{60s}{1min} \cdot \frac{60min}{1h} \simeq 233'172\ Km/h}$$

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b) ¿ tiempo en que la flecha es propulsada?

$F = -k r$

$m\ddot{r} = -kr \longrightarrow$ oscilador armónico $\quad \omega = \sqrt{\frac{k}{m}}$

$r(t) = r_i \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

$r(t_{salida}) = 0 = 0'72\,m \cdot \cos\left(\sqrt{\frac{k}{m}}\, t_{salida}\right) \Rightarrow \cos\left(\sqrt{\frac{k}{m}}\, t_s\right) = 0$

$\Downarrow$

$\sqrt{\frac{k}{m}}\, t_s = (2n+1)\frac{\pi}{2} \xrightarrow[n=0]{1^{a}\text{ sol.}} \sqrt{\frac{k}{m}}\, t_s = \frac{\pi}{2}$

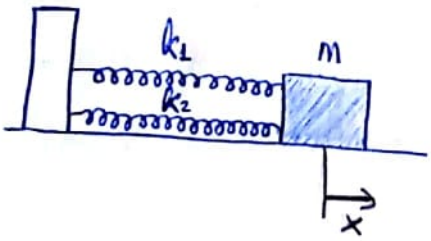
$t_s = \frac{\pi}{2}\sqrt{\frac{m}{k}}$

$$\boxed{t_s = \frac{\pi}{2}\sqrt{\frac{0'023\,kg}{186'2\,N/m}} \simeq 0'017\ s}$$

3.4

Ecuaciones del movimiento para los sistemas:

(a)



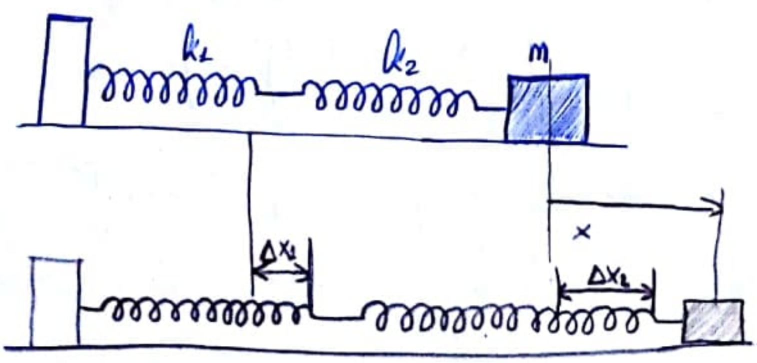
$m\ddot{x} = -k_1 x - k_2 x$

$$\boxed{m\ddot{x} = -(k_1 + k_2)\,x}$$

Resortes acoplados en paralelo:

$$\boxed{k_{eq} = k_1 + k_2}$$

(b)



$x = \Delta x_1 + \Delta x_2$

$$\begin{cases} \Delta x_1 = \dfrac{k_2}{k_1 + k_2}\, x \\ \Delta x_2 = \dfrac{k_1}{k_1 + k_2}\, x \end{cases}$$

$$m\ddot{x} = -k_2\,\Delta x_2 = -k_2\,\frac{k_1}{k_1+k_2}\,x = -\frac{k_1 k_2}{k_1+k_2}\,x$$

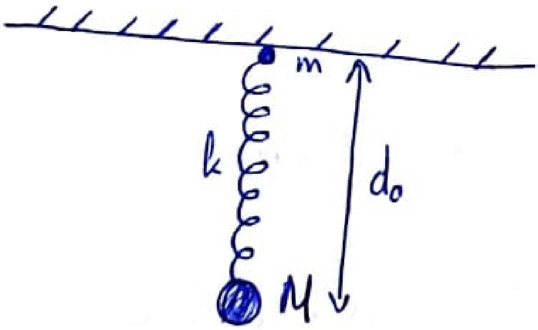
$$k = \frac{k_1 k_2}{k_2+k_2} \longrightarrow \boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}}$$

Resortes acoplados en serie

(3.5) Resorte ideal de
- masa despreciable
- constante k
- longitud natural l_0

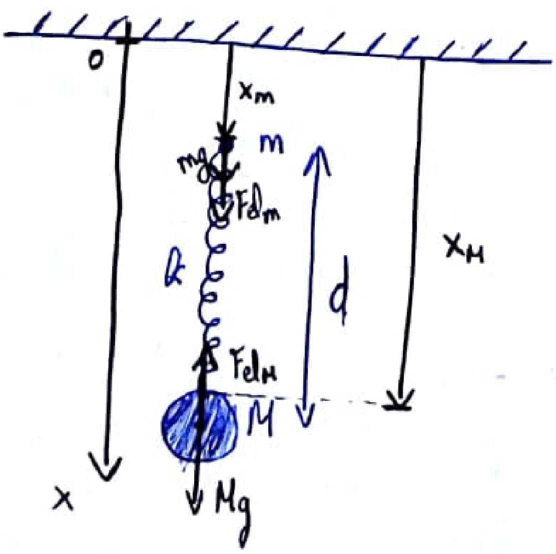
Unido en sus extremos a 2 masas m y M

m → techo → muelle suspendido → se suelta en $t=0$



$$\boxed{d = l_0 + \Delta x} \longrightarrow \Delta x = d - l_0$$
$$\Delta x = x_M - x_m - l_0$$

$$\begin{cases} m\ddot{x}_m = mg + k\Delta x = mg + k(x_M - x_m - l_0) & [1] \\ M\ddot{x}_M = Mg - k\Delta x = Mg - k(x_M - x_m - l_0) & [2] \end{cases}$$



$\boxed{[1] + [2]}$

$$m\ddot{x}_m + M\ddot{x}_M = (m+M)g$$

$$\frac{m\ddot{x}_m + M\ddot{x}_M}{m+M} = g$$

$$\frac{d^2}{dt^2}\left(\frac{m x_m + M x_M}{m+M}\right) = g$$

$$\boxed{\frac{d^2}{dt^2}(x_{CM}) = g} \longrightarrow$$ El CDM cae con aceleración g

$$\frac{d^2}{dt^2} x_{CM} = g \,, \quad \frac{d}{dt} x_{CM} = gt + \cancel{\dot{x}_{CM0}}^{\,0}$$

$$\boxed{x_{CM}} = \frac{1}{2} g t^2 + x_{CM0} = \boxed{\frac{1}{2} g t^2 + \frac{M d_0}{M+m}}$$

$$\boxed{\circledast \; d_0 = x_M^0 - x_m^0}$$

$$\boxed{[2]\, m - [1]\, M}$$

$$\begin{aligned} & m M \ddot{x}_M = m M g - k m (x_M - x_m - l_0) \\ - & \left(m M \ddot{x}_m = m M g + k M (x_M - x_m - l_0) \right) \\ \hline & m M (\ddot{x}_M - \ddot{x}_m) = - k (x_M - x_m - l_0)(m + M) \end{aligned}$$

$$\ddot{x}_M - \ddot{x}_m = - k \left(\frac{m+M}{mM} \right) (x_M - x_m - l_0)$$

Def: $\left.\begin{aligned} z &\equiv x_M - x_m - l_0 \\ \dot{z} &= \dot{x}_M - \dot{x}_m \\ \ddot{z} &= \ddot{x}_M - \ddot{x}_m \end{aligned}\right\}$ $\boxed{\ddot{z} = - k \left(\frac{m+M}{mM} \right) z}$

oscilador armónico con

$$\omega = \sqrt{k \frac{(m+M)}{mM}}$$

$z(t) = A \cos(\omega t - \delta)$ $\longleftarrow$ la distancia entre las masas oscila con frecuencia ω

$\boxed{t=0}$ $\dot{z}(t=0) = 0$

$\boxed{t=0}$ $z(t=0) = x_M^0 - x_m^0 - l_0 = d_0 - l_0$

$\dot{z}(t) = - A\omega \sin(\omega t - \delta)$, $\dot{z}(0) = - A\omega \sin(-\delta) = A\omega \sin\delta$

$A\omega \sin\delta = 0 \Rightarrow \sin\delta = 0 \Rightarrow \boxed{\delta = 0}$

$z(t) = A \cos(\omega t)$, $z(0) = A = d_0 - l_0$

$z(t) = (d_0 - l_0)\cos(\omega t) \Rightarrow \boxed{x_M - x_m = (d_0 - l_0)\cos\omega t + l_0}$

$$(a) \quad x_{CM}(t) = \frac{1}{2} g t^2 + \frac{M d_0}{M+m} = \frac{m\, x_m(t) + M\, x_M(t)}{m + M}$$

$$(b) \quad x_M(t) - x_m(t) = (d_0 - \ell_0) \cos \omega t + \ell_0$$

$$(a) \quad m\, x_m + M\, x_M = \frac{m+M}{2} g t^2 + M d_0$$

$$x_M = \frac{m+M}{2M} g t^2 + d_0 - \frac{m}{M} x_m$$

$$(b) \quad x_m = x_M - \ell_0 - (d_0 - \ell_0) \cos \omega t$$

$$x_m = \frac{m+M}{2M} g t^2 + (d_0 - \ell_0) - (d_0 - \ell_0) \cos \omega t - \frac{m}{M} x_m$$

$$x_m \left(1 + \frac{m}{M}\right) = \frac{m+M}{2M} g t^2 + (d_0 - \ell_0)(1 - \cos \omega t)$$

$$x_m \left(\frac{M+m}{M}\right) = \frac{m+M}{M} \frac{1}{2} g t^2 + (d_0 - \ell_0)(1 - \cos \omega t)$$

$$\boxed{x_m(t) = \frac{1}{2} g t^2 + \frac{M}{m+M} (d_0 - \ell_0)(1 - \cos \omega t)}$$

$$x_M(t) = \frac{m+M}{M} \frac{1}{2} g t^2 + d_0 - \frac{m}{M} \left[\frac{1}{2} g t^2 + \frac{M}{m+M} (d_0 - \ell_0)(1 - \cos \omega t)\right]$$

$$x_M(t) = \frac{m+M}{M} \frac{1}{2} g t^2 + d_0 - \frac{m}{M} \frac{1}{2} g t^2 + \frac{m}{m+M} (d_0 - \ell_0)(1 - \cos \omega t)$$

$$\boxed{x_M(t) = \frac{1}{2} g t^2 + \frac{m}{m+M} (d_0 - \ell_0)(1 - \cos \omega t) + d_0}$$

$$\Big\} \; F_{el} = k \Delta x_0 = Mg$$

$$\Delta x_0 = \frac{Mg}{k}$$

$$d_0 = \ell_0 + \Delta x_0 = \ell_0 + \frac{Mg}{k}$$

$$\boxed{d_0 = \ell_0 + \frac{Mg}{k}}$$

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3 Ejs

Solución :

$$x_m(t) = \frac{1}{2} g t^2 + \frac{M^2 g}{(m+M)k}\left[1 - \cos(\omega t)\right]$$

$$x_M(t) = \frac{1}{2} g t^2 + \frac{M m g}{(M+m)k}\left[1 - \cos(\omega t)\right] + l_0 + \frac{Mg}{k}$$

Donde $\omega = \sqrt{k \frac{(m+M)}{mM}}$

3.6 Densidad de probabilidad de que un oscilador armónico esté en la posición x : $x(t) = A\cos(\omega t - \delta)$

Sea $p(x)$ la función densidad de probabilidad.

$$\int_x^{x+\Delta x} p(u)\,du \longrightarrow \text{Probabilidad de estar en el intervalo } [x, x+\Delta x]$$

(37) Cierre automático de una puerta ⟶ oscilador amortiguado

(resorte con un émbolo que proporciona un rozamiento proporcional a la velocidad).

puerta + cierre ⟶ $m = 10\,kg$

$k = 10\,N/cm$

$\boxed{F_{roz} = -bv}$

a) b para amortiguamiento crítico?

Amortiguamiento crítico ⟶ $\gamma_c^2 = \omega_0^2$ frecuencia libre

→ Calculamos la frecuencia libre del oscilador como

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \omega_0^2 = \frac{k}{m}$$

→ El coeficiente de amortiguamiento se define como: $\gamma = \frac{b}{2m}$,

$$\gamma^2 = \frac{b^2}{4m^2}$$

$$\frac{b^2}{4m^2} = \frac{k}{m}, \quad b = \sqrt{4mk} = 2\sqrt{mk}$$

$$b_c = 2\sqrt{10\,kg \cdot 10\,N/cm \cdot \frac{1\,cm}{10^{-2}\,m}} = 200\ kg/s$$

b) $b = 2000\,kg/s > b_c \Rightarrow \gamma > \gamma_c = \omega_0$ ⟶ Sobreamortiguado

Puerta { cerrada ⟶ $\Delta x_c = 0'5\ cm = 5 \cdot 10^{-3}\,m$
abierta ⟶ $\Delta x_a = 0'1\,m$

¿ v_f al cerrar ? ¿ t ? $\quad x(t) = e^{-\gamma t}\left[A_1 e^{\omega_1 t} + A_2 e^{-\omega_1 t}\right]$

$$x(t) = A_1\, e^{(\omega_1-\gamma)t} + A_2\, e^{(-\omega_1-\gamma)t} = A_1\, e^{\gamma_1 t} + A_2\, e^{\gamma_2 t}$$

$$\omega_1^2 = \gamma^2 - \omega_0^2, \quad \omega_1 - \gamma \equiv \gamma_1 \quad -\omega_1 - \gamma \equiv \gamma_2$$

$$\dot{x}(t) = A_1\gamma_1\, e^{\gamma_1 t} + A_2\gamma_2\, e^{\gamma_2 t}$$

$x(t=0) = A_1 + A_2 = 0'1m \Rightarrow A_2 = 0'1m - A_1$

$\dot{x}(t=0) = A_1\gamma_1 + A_2\gamma_2 = 0$, $A_1\gamma_1 + 0'1m\gamma_2 - A_1\gamma_2 = 0$

$\gamma = \frac{b}{2m}$, $\gamma = \frac{2000Kg/s}{2\ 10Kg} = 100\,s^{-1}$ $\qquad A_1(\gamma_1-\gamma_2) = -0'1m\,\gamma_2$

$\omega_0^2 = \frac{k}{m}$, $\omega_0^2 = \frac{1000N/m}{10Kg} = 100\,s^{-2}$ $\qquad A_1 = \frac{0'1m\,\gamma_2}{\gamma_2-\gamma_1}$

$\omega_1^2 = \gamma^2 - \omega_0^2 = (100s^{-1})^2 - 100s^{-2} = 9900\,s^{-2}$

$\gamma_1 = \omega_1 - \gamma = \sqrt{9900s^{-2}} - 100\,s^{-1} \simeq -0'5\ s^{-1}$ $\qquad A_1 \simeq 0'10025\ m$

$\gamma_2 = -\omega_1 - \gamma \simeq -199'5\ s^{-1}$ $\qquad A_2 \simeq -2'5\cdot 10^{-4}m$

$x(tf) = 0'10025m\ e^{-0'5s^{-1}tf} \underbrace{- 2'5\ 10^{-4}m\ e^{-199'5\,s^{-1}tf}}_{\text{despreciable}} = 5\cdot10^{-3}m$

$e^{-0'5s^{-1}tf} \simeq 0'04988 \rightarrow -0'5s^{-1}tf = \ln 0'04988$

$\boxed{tf} = \frac{-\ln 0'04988}{0'5}\,s \simeq \boxed{6\,s}$

$\boxed{\dot{x}(tf=6s)} = 0'10025m\cdot(-0'5\,s^{-1})\ e^{-0'5s\cdot 6s} - 2'5\cdot10^{-4}m\,(-199'5s^{-1})\ e^{-199'5s^{-1}6s} \simeq$

$\simeq -2'5\cdot10^{-3}\ m/s = \boxed{-0'25\ cm/s}$

c) γ para que tarde el doble? Qué tipo de oscilador sería?

$tf = 12s$ $\qquad x(12s) = 5\cdot10^{-3}m$ $\qquad x(0s) = 0'1m$

Tarda más $\longrightarrow$ Mayor rozamiento $\rightarrow$ el término que iba con $e^{-\gamma_2 t}$ ahora se va a cero aún más rápido

Aproximamos $\qquad x(t) \simeq A_1 e^{\gamma_1 t}$

$x(0s) \simeq A_1 \simeq 0'1m$

$x(12s) = 0'1m\ e^{\gamma_1\cdot 12s} = 5\ 10^{-3}m$, $\qquad \gamma_1\ 12s = \ln 0'05$

$\gamma_1 \simeq -0'2496\,s^{-1}$

$\gamma_1 = \omega_1 - \gamma = \sqrt{\gamma^2 - \omega_0^2} - \gamma$

$$\gamma_1 + \gamma = \sqrt{\gamma^2 - \omega_0^2} \quad , \quad \gamma_1^2 + \gamma^2 + 2\gamma\gamma_1 = \gamma^2 - \omega_0^2$$

$$\gamma_1^2 + 2\gamma\gamma_1 = -\omega_0^2$$

$$\gamma = \frac{-\gamma_1^2 - \omega_0^2}{2\gamma_1} \quad \longleftarrow \quad -\gamma_1^2 - \omega_0^2 = 2\gamma\gamma_1$$

$$\gamma \simeq \frac{-(-0'2496\,s^{-1})^2 - 100\,s^{-2}}{2 \cdot (-0'2496\,s^{-1})} \simeq \boxed{200\ s^{-1}} \longrightarrow$$ Oscilador sobreamortiguado

$$\gamma^2 > \omega_0^2$$

d) ¿Cómo varían la velocidad y el tiempo de cierre si cambiamos la puerta por una que pese la mitad?

$$x(t) = e^{-\gamma t}\left[A_1\, e^{\sqrt{\gamma^2-\omega_0^2}\,t} + A_2\, e^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right] =$$

$$= e^{-\frac{b}{2m}t}\left[A_1\, e^{\sqrt{\frac{b^2}{4m^2}-\frac{k}{m}}\,t} + A_2\, e^{-\sqrt{\frac{b^2}{4m^2}-\frac{k}{m}}\,t}\right]$$

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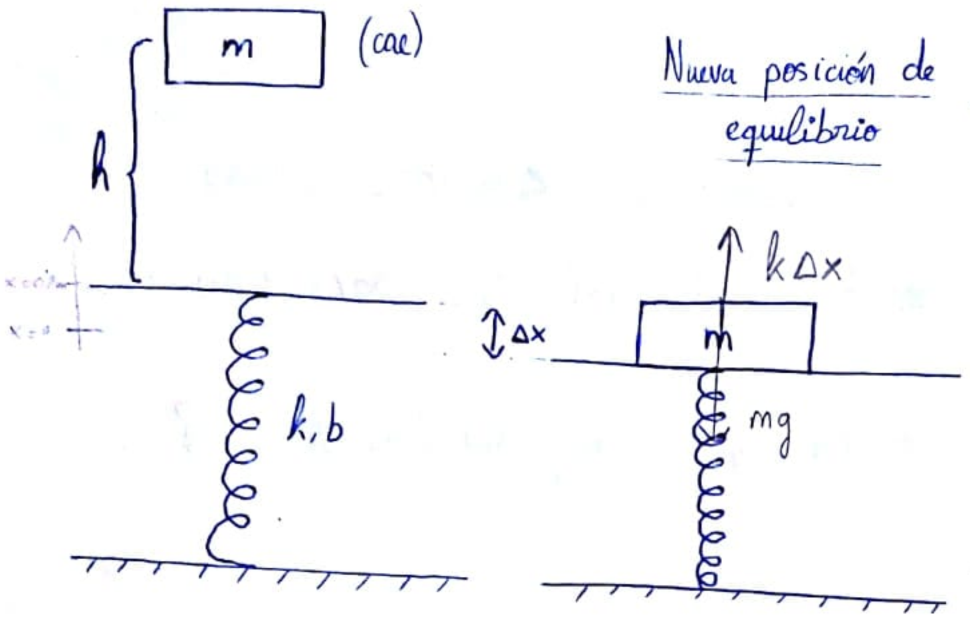
(38) $m = 1000\,Kg$ $\quad h = 10\,m$ $\quad \Delta x = 0'2\,m$ $\quad g = 9'8\,m/s^2$



Nueva posición de equilibrio → Diseñado de forma que la alcance lo más rápido posible → Amortiguamiento crítico

$$\gamma_c^2 = \omega_0^2$$

$$\frac{b^2}{4m^2} = \frac{k}{m}$$

a) b, k?

Equlibrio $\Longrightarrow$ $k\,\Delta x = mg$, $\quad k = \dfrac{mg}{\Delta x}$, $\quad \boxed{k = 49000\,N/m}$

$b^2 = 4km$, $\quad b = 2\sqrt{km}$, $\quad \boxed{b = 14000\,Kg/s}$

b) $x(t)$ a partir del impacto? $\quad x(t) = (A + Bt)\,e^{-\gamma t}$

c. i. $\longrightarrow$ $\underbrace{\begin{array}{l} x(0) = 0'2\,m \\ \dot{x}(0) = v_0 \longrightarrow\ ? \end{array}}$

Conservación de la energía en la caída:

$$mgh = \frac{1}{2} m v_0^2, \quad v_0 = \sqrt{2gh}$$

$x(0) = A = 0'2\,m \Rightarrow \boxed{A = 0'2\,m}$

$\dot{x}(t) = B e^{-\gamma t} + (A + Bt)(-\gamma)\,e^{-\gamma t} = e^{-\gamma t}\left[B - \gamma(A + Bt)\right]$

$\dot{x}(0) = B - \gamma A = \sqrt{2gh}$, $\quad B = \sqrt{2gh} + \dfrac{b}{2m} A$

$$\boxed{B = 15'4\,m/s}$$

$$\boxed{x(t) = \left(0'2\,m + 15'4\,m/s\; t\right) e^{-7s^{-1}t}}$$ $[t] = s$

$\gamma = \frac{b}{2m}$, $\gamma = 7s^{-1}$

3.9 Oscilador $\longrightarrow$ periodo libre $\tau_0 = 1'000\,s$ (ω_0)

$\xrightarrow{\text{amortiguamiento}}$ nuevo periodo $\tau_1 = 1'001\,s$ (ω_1)

¿ γ? ¿Cuánto disminuye la amplitud tras 10 ciclos?

Oscilador infraamortiguado : $\gamma^2 < \omega_0^2$

$\omega_1^2 = \omega_0^2 - \gamma^2$ $\Big\}$ $\boxed{x(t) = A\, e^{-\gamma t} \cos(\omega_1 t - \delta)}$

$\omega_0 = \frac{2\pi}{\tau_0}$

$\omega_1 = \frac{2\pi}{\tau_1}$

$$\gamma^2 = \omega_0^2 - \omega_1^2$$

$$\gamma^2 = \left(\frac{2\pi}{\tau_0}\right)^2 - \left(\frac{2\pi}{\tau_1}\right)^2$$

$$\gamma = \sqrt{4\pi^2\left(\frac{1}{\tau_0^2} - \frac{1}{\tau_1^2}\right)} = 2\pi\sqrt{\frac{\tau_1^2 - \tau_0^2}{\tau_0^2\tau_1^2}}$$

$$\gamma \simeq 0'045\,\omega_0 \longleftarrow \gamma = \frac{2\pi}{\tau_0}\,\frac{\sqrt{\tau_1^2 - \tau_0^2}}{\tau_1}$$

$$\boxed{\gamma = 0'283\ s^{-1}}$$

$t_0 = 0 \longrightarrow$ amplitud A

$t_1 = 10\,\tau_1 = 10'010\,s$

amplitud $\longrightarrow A(t) = A\, e^{-\gamma t}$

$$A(t_1) = A\, e^{-0'283\, t_1}$$

$$\boxed{\frac{A(t_1)}{A} = e^{-0'283 \cdot 10'010\,s} \simeq 0'06}$$

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3.10 Oscilador armónico amortiguado forzado

$$F(t) = f e^{-\mu t} \qquad \gamma^2 > \omega_0^2 \rightarrow \text{sobreamortiguado}$$

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t)$$

→ Ec homogénea: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

$$x_h(t) = C_1 e^{(-\gamma + \sqrt{\gamma^2-\omega_0^2})t} + C_2 e^{(-\gamma - \sqrt{\gamma^2-\omega_0^2})t}$$

Def: $\left.\begin{array}{l} \gamma_1 \equiv \gamma - \sqrt{\gamma^2-\omega_0^2} \\ \gamma_2 \equiv \gamma + \sqrt{\gamma^2-\omega_0^2} \end{array}\right\} \rightarrow \boxed{x_h(t) = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}}$

→ Solución particular: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{f}{m} e^{-\mu t}$

$\left.\begin{array}{l} x_p(t) = B e^{\lambda t} \\ \dot{x}_p(t) = B\lambda e^{\lambda t} \\ \ddot{x}_p(t) = B\lambda^2 e^{\lambda t} \end{array}\right\} \longrightarrow B\lambda^2 e^{\lambda t} + 2\gamma\lambda B e^{\lambda t} + \omega_0^2 B e^{\lambda t} = \frac{f}{m} e^{-\mu t}$

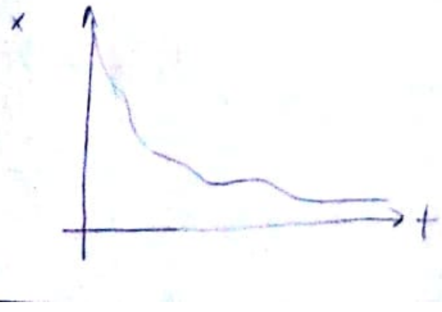
$$B(\lambda^2 + 2\gamma\lambda + \omega_0^2) e^{\lambda t} = \frac{f}{m} e^{-\mu t}$$

$\boxed{\circledast\ \lambda = -\mu}$ $\longrightarrow$ $B(\overset{\mu^2}{\lambda^2} + 2\gamma \overset{-\mu}{\lambda} + \omega_0^2) = \frac{f}{m}$

$$\boxed{B = \frac{f/m}{\mu^2 - 2\gamma\mu + \omega_0^2}}$$

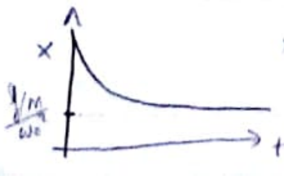
$$\boxed{x_p(t) = \frac{f/m}{\mu^2 - 2\gamma\mu + \omega_0^2} e^{-\mu t}}$$

$$x(t) = x_h(t) + x_p(t) = \underbrace{C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}}_{\text{Término transitorio}} + \frac{f/m}{\mu^2 - 2\gamma\mu + \omega_0^2} e^{-\mu t}$$



$\circledast$ Si $\mu = 0 \Rightarrow F(t) = f = \text{cte}$

$$x(t) = \underbrace{C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}}_{t \text{ trans}} + \underbrace{\frac{f/m}{\omega_0^2}}_{t \text{ estacionario cte}}$$



(3.11) Osciladør forzado $\longrightarrow$ máxima respuesta A^2 $\longrightarrow$ probar que es igual a la mitad de su valor máximo cuando $\omega \simeq \omega_0 \pm \gamma$
ocurre cuando $\omega \simeq \omega_0$ para $\gamma \ll \omega_0$
$\hookrightarrow$ anchura 2γ

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$$A^2(\omega) = \frac{F_0^2/m^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2} \qquad A^2_{máx} = A^2(\omega_0) = \frac{F_0^2/m^2}{4\gamma^2\omega_0^2}$$

$$A^2(\omega_0 \pm \gamma) = \frac{F_0^2/m^2}{(\omega_0^2-(\omega_0\pm\gamma)^2)^2+4\gamma^2(\omega_0\pm\gamma)^2} =$$

$$= \frac{F_0^2/m^2}{\left[\omega_0^2(1-(1\pm\gamma/\omega_0)^2)\right]^2+4\gamma^2\omega_0^2(1\pm\gamma/\omega_0)^2} \simeq$$

(*) $\underbrace{\gamma \ll \omega_0}$ $\Downarrow$ $\underbrace{\frac{\gamma}{\omega_0} \ll 1}$ Despreciamos $\frac{\gamma}{\omega_0}$ frente a 1

$$\simeq \frac{F_0^2/m^2}{\left[\omega_0^2\left(1-1\mp 2\gamma/\omega_0-\frac{\gamma^2}{\omega_0^2}\right)\right]^2+4\gamma^2\omega_0^2(1\pm\gamma/\omega_0)^2} \simeq$$

$$\simeq \frac{F_0^2/m^2}{(\mp\omega_0^2\, 2\gamma/\omega_0)^2+4\gamma^2\omega_0^2(1\pm\gamma/\omega_0)^2} \overset{(*)}{\simeq}$$

$$\simeq \frac{F_0^2/m^2}{4\gamma^2\omega_0^2+4\gamma^2\omega_0^2} = \frac{F_0^2/m^2}{8\gamma^2\omega_0^2} = \boxed{\frac{1}{2}A^2_{max}} \quad \text{q. e. d.}$$

(3.12) a) c.i. de un oscilador armónico infraamortiguado de forma que alcance inmediatamente la situación estacionaria al ser forzado con $F(t) = F_0 \cos\omega t$ t_0

$$x_1(t) = A e^{-\gamma t}\cos(\omega_1 t+\theta) \qquad \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\cos(\omega t-\delta) = x_2(t)$$

(*) Continuidad de la posición en t_0 $\rightsquigarrow$ $A e^{-\gamma t_0}\cos(\omega_1 t_0+\theta) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\cos(\omega t$

$$\dot{x}_1(t) = -A\gamma e^{-\gamma t}\cos(\omega_1 t+\theta) - A\omega_1 e^{-\gamma t}\sin(\omega_1 t+\theta) =$$

$$= -A e^{-\gamma t}\left[\gamma\cos(\omega_1 t+\theta) + \omega_1\sin(\omega_1 t+\theta)\right]$$

$$\dot{x}_2(t) = -\frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\,\omega\sin(\omega t-\delta)$$

⊛ Continuidad de la velocidad en t_0

$$-Ae^{-\gamma t_0}\left[\gamma\cos(\omega_1 t_0+\theta)+\omega_1\sin(\omega_1 t_0+\theta)\right] = \frac{-F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\cdot\omega\cdot\sin(\omega t_0-\delta)$$

Cociente de ambas ecuaciones

$$\frac{-Ae^{-\gamma t_0}\left[\gamma\cos(\omega_1 t_0+\theta)+\omega_1\sin(\omega_1 t_0+\theta)\right]}{Ae^{-\gamma t_0}\cos(\omega_1 t_0+\theta)} = \frac{\frac{-F_0/m}{\sqrt{(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2}}\cdot\omega\cdot\sin(\omega t_0-\delta)}{\frac{F_0/m}{\sqrt{(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2}}\cdot\cos(\omega t_0-\delta)}$$

$$\gamma+\omega_1\tan(\omega_1 t_0+\theta) = \omega\tan(\omega t_0-\delta)$$

Suponemos $t_0=0$:

$$\gamma+\omega_1\tan\theta = \omega\tan(-\delta) \qquad \tan\delta = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

$$\tan\theta = \frac{-\omega\tan\delta-\gamma}{\omega_1} = -\frac{\gamma}{\omega_1}\left(1+\frac{2\omega^2}{\omega_0^2-\omega^2}\right) = -\frac{\gamma}{\omega_1}\left(\frac{\omega_0^2-\omega^2+2\omega^2}{\omega_0^2-\omega^2}\right)$$

$$\boxed{\tan\theta = -\frac{\gamma}{\omega_1}\left(\frac{\omega^2+\omega_0^2}{\omega_0^2-\omega^2}\right)} \qquad A\cos\theta = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\cos\delta$$

$$A = \frac{F_0}{m}\,\frac{\cos\delta/\cos\theta}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} = \frac{F_0}{m}\,\frac{\sqrt{\frac{1+\tan^2\theta}{1+\tan^2\delta}}}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} = \frac{F_0}{m}\,\frac{\sqrt{\frac{1+\frac{\gamma^2}{\omega_1^2}\left(\frac{\omega^2+\omega_0^2}{\omega_0^2-\omega^2}\right)^2}{1+\frac{4\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2}}}}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} =$$

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}$$

$$= \frac{F_0}{m}\,\frac{\sqrt{\frac{\omega_0^2-\omega^2+\frac{\gamma^2(\omega^2+\omega_0^2)^2}{\omega_0^2-\gamma^2}}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

? No da

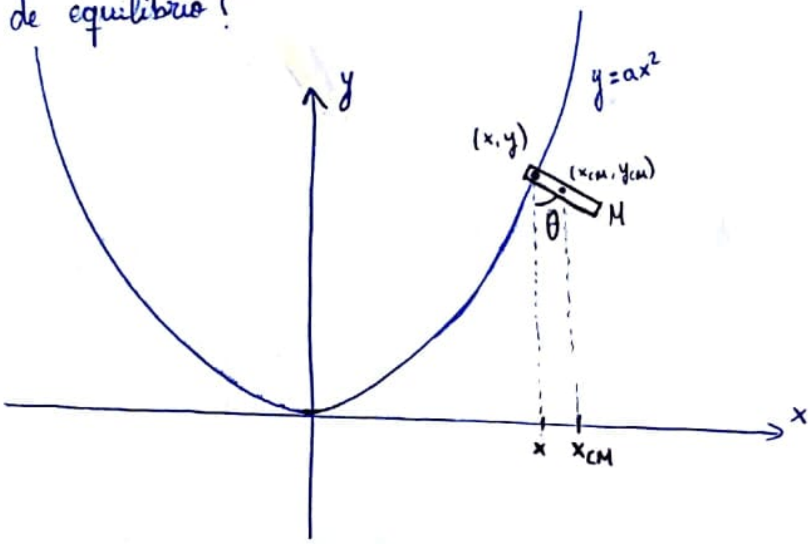
3.14 Vara homogénea $\longrightarrow$ masa M, longitud L

$\boxed{I = \frac{1}{12} M L^2}$

$\hookrightarrow$ Suspendida de uno de sus extremos de un alambre parabólico $y = a x^2$ $(a > 0)$

El extremo resbala sin rozamiento por el alambre

¿Frecuencias propias de las pequeñas oscilaciones alrededor de la posición de equilibrio?



CM $\longrightarrow$ punto medio

3 coordenadas	$(x,y), \theta$	
1 ligadura	$y = a x^2$	

$\Rightarrow 2\,g\,\ell \Rightarrow$ 2 coord. gen. $\quad q_1 = x \quad q_2 = \theta$

$$\boxed{T = T_{\substack{\text{traslación} \\ CM}} + T_{\substack{\text{rotación en} \\ \text{torno al CM}}}}$$ (2° Th König)

$$T_{\text{trasl CM}} = \frac{1}{2} M \left(\dot{x}_{CM}^2 + \dot{y}_{CM}^2 \right)$$

$$\begin{cases} x_{CM} = x + \frac{L}{2}\sin\theta \\ y_{CM} = y - \frac{L}{2}\cos\theta = a x^2 - \frac{L}{2}\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x}_{CM} = \dot{x} + \frac{L}{2}\dot{\theta}\cos\theta \\ \dot{y}_{CM} = 2 a x \dot{x} + \frac{L}{2}\dot{\theta}\sin\theta \end{cases}$$

$$T_{trasl\,CM} = \frac{1}{2} M \left[\left(\dot{x} + \frac{L}{2}\dot{\theta}\cos\theta\right)^2 + \left(2a\dot{x}x + \frac{L}{2}\dot{\theta}\sin\theta\right)^2 \right] =$$

$$= \frac{M}{2} \left[\dot{x}^2 + \frac{L^2}{4}\dot{\theta}^2\cos^2\theta + L\dot{\theta}\dot{x}\cos\theta + 4a^2x^2\dot{x}^2 + \frac{L^2}{4}\dot{\theta}^2\sin^2\theta + \right.$$

$$\left. + 2L\dot{\theta}\dot{x}\,a\,x\,\sin\theta \right] =$$

$$= \frac{M}{2}\left[(1+4a^2x^2)\dot{x}^2 + \frac{L^2}{4}\dot{\theta}^2 + L(\cos\theta + 2ax\sin\theta)\dot{x}\dot{\theta} \right]$$

$$T_{rot} = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2 = \frac{1}{24} ML^2\dot{\theta}^2$$

$$T = \frac{M}{2}\left[(1+4a^2x^2)\dot{x}^2 + L(\cos\theta + 2ax\sin\theta)\dot{x}\dot{\theta} \right] + \frac{ML^2\dot{\theta}^2}{8} + \frac{ML^2\dot{\theta}^2}{24} =$$

$$= \frac{M}{2}\left[(1+4a^2x^2)\dot{x}^2 + L(\cos\theta + 2ax\sin\theta)\dot{x}\dot{\theta} + \frac{L^2\dot{\theta}^2}{3} \right]$$

$$V = Mg\,y_{cm} = Mg\left(y - \frac{L}{2}\cos\theta\right) = Mg\left(ax^2 - \frac{L}{2}\cos\theta\right)$$

$(V=0,\ y=0)$

$$\circledast \quad T = \frac{1}{2}\sum_{j,k=1}^{2} m_{jk}\dot{q}_j\dot{q}_k \quad , \quad V = \frac{1}{2}\sum_{j,k=1}^{2} A_{jk}\,q_j\,q_k$$

⇝ Oscilaciones pequeñas en torno al punto de equilibrio (x_0, θ_0):

$$\left.\frac{\partial V}{\partial q_k}\right|_{q_{k_0}} = 0$$

$$\frac{\partial V}{\partial x} = 2Mgax$$

$$\left.\frac{\partial V}{\partial x}\right|_{x_0} = 0 \implies \boxed{x_0 = 0}$$

$$\frac{\partial V}{\partial \theta} = \frac{MgL}{2}\sin\theta$$

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0} = \frac{MgL}{2}\sin\theta_0 = 0 \implies \boxed{\theta_0 = 0}$$

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Osculaciones pequeñas $\longrightarrow$ $\sin\theta \simeq \theta$

$\boxed{\theta \simeq 0}$ $\qquad \cos\theta \simeq 1 - \frac{1}{2}\theta^2$

$$T \simeq \frac{M}{2}\left[(1+4a^2x^2)\dot{x}^2 + L\dot{x}\dot{\theta}\left(1-\frac{1}{2}\theta^2+2ax\theta\right) + \frac{L^2}{3}\dot{\theta}^2\right] \simeq$$

$\circledast$ $x \simeq 0$, $\theta \simeq 0$

$$\simeq \frac{M}{2}\left[\dot{x}^2 + L\dot{x}\dot{\theta} + \frac{L^2}{3}\dot{\theta}^2\right] =$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}ML\dot{x}\dot{\theta} + \frac{1}{2}\frac{ML^2}{3}\dot{\theta}^2 =$$

$$= \frac{1}{2}m_{11}\dot{x}^2 + \underbrace{\frac{1}{2}m_{12}\dot{x}\dot{\theta} + \frac{1}{2}m_{21}\dot{\theta}\dot{x}} + \frac{1}{2}m_{22}\dot{\theta}^2 =$$

$$= \frac{1}{2}m_{11}\dot{x}^2 + m_{12}\dot{x}\dot{\theta} + \frac{1}{2}m_{22}\dot{\theta}^2$$

$$\left.\begin{aligned} m_{11} &= M \\ m_{21} = m_{12} &= \frac{1}{2}ML \\ m_{22} &= \frac{ML^2}{3} \end{aligned}\right\} \Rightarrow \{m\} = \begin{pmatrix} M & \frac{1}{2}ML \\ \frac{1}{2}ML & \frac{ML^2}{3} \end{pmatrix}$$

$$V = Mg\left(ax^2 - \frac{L}{2}\cos\theta\right) \simeq Mg\left(ax^2 - \frac{L}{2}\left(1-\frac{\theta^2}{2}\right)\right) =$$

$$= Mgax^2 + \frac{1}{4}MgL\theta^2 - MgL\frac{1}{2}$$

$$A_{jk} = \frac{\partial^2 V}{\partial q_k \partial q_j}$$

$$\frac{\partial V}{\partial x} = 2Mgax \qquad \frac{\partial V}{\partial \theta} = \frac{1}{2}MgL\theta$$

$$\frac{\partial^2 V}{\partial x^2} = 2Mga \qquad \frac{\partial^2 V}{\partial \theta^2} = \frac{1}{2}MgL$$

$$\underbrace{\frac{\partial^2 V}{\partial x\partial\theta} = \frac{\partial^2 V}{\partial x\partial\theta} = 0}$$

$$\{A\} = \begin{pmatrix} 2Mga & 0 \\ 0 & \frac{1}{2}MgL \end{pmatrix}$$

Ecuación secular: $\det\left[\{A\} - \omega^2\{m\}\right] = 0$

$$\det\begin{pmatrix} 2Mga - \omega^2 \cancel{M} & -\frac{1}{2}ML\omega^2 \\ -\frac{1}{2}ML\omega^2 & \frac{1}{2}MgL - \frac{ML^2}{3}\omega^2 \end{pmatrix} = 0$$

$$(2ga - \omega^2)\left(\frac{1}{2}gL - \frac{L^2\omega^2}{3}\right) - \frac{1}{4}L^2\omega^4 = 0$$

$$g^2 a - \frac{2}{3}L\omega^2 ga - \frac{1}{2}g\omega^2 + \frac{L\omega^4}{3} - \frac{1}{4}L\omega^4 = 0$$

$$g^2 a - \left(\frac{1}{2}g + \frac{2}{3}gaL\right)\omega^2 + \frac{1}{12}L\omega^4 = 0$$

Ec. bicuadrada

$$L\omega^4 - 6g\left(1 + \frac{4}{3}aL\right)\omega^2 + 12g^2a = 0$$

$$\omega_{1,2}^2 = \frac{\overset{3}{\cancel{6}}g\left(1+\frac{4}{3}aL\right) \pm \sqrt{\overset{9}{\cancel{36}}g^2\left(1+\frac{4}{3}aL\right)^2 - \overset{12}{\cancel{48}}Lg^2a}}{\cancel{2}L}$$

$$\omega_{1,2}^2 = \frac{3g}{L}\left(1+\frac{4}{3}aL\right) \pm \frac{g}{L}\sqrt{9\left(1+\frac{4}{3}aL\right)^2 - 12La}$$

$$\omega_{1,2}^2 = \frac{g}{L}(3+4aL) \pm \frac{g}{L}\,3\left(1+\frac{4}{3}aL\right)\sqrt{1 - \frac{12La}{9\left(1+\frac{4}{3}aL\right)^2}}$$

$$\boxed{\omega_{1,2}^2 = \frac{g}{L}(3+4aL)\left[1 \pm \sqrt{1 - \frac{4La}{3\left(1+\frac{4}{3}aL\right)^2}}\right]}$$

⊛ Para $a \to 0$, $\omega_1 \simeq 0$

$\omega_2 \simeq \sqrt{6}\sqrt{\frac{g}{L}}$

(3.15) Barra homogénea $\longrightarrow m_1, L$

$\underbrace{\text{Barra homogénea}}$ colgada del techo por $\underbrace{\text{dos hilos}}$ unidos a sus extremos

longitud a
peso despreciable

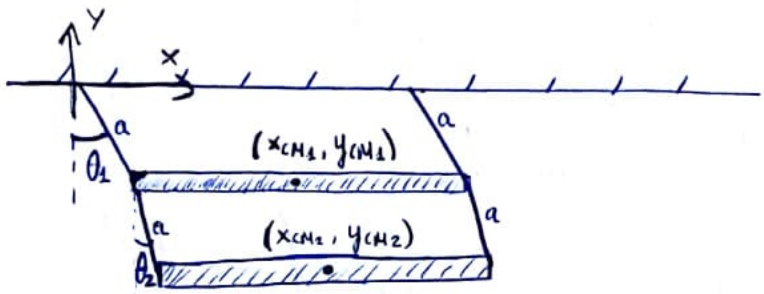
$\circledast$ Siempre horizontales

¿ Frecuencias de pequeñas oscilaciones ?

Barra homogénea $\longrightarrow m_2, L$

$\underbrace{\text{Barra homogénea}}$ colgada de la primera mediante $\underbrace{\text{dos hilos}}$

longitud a,
peso despreciable



2 Coordenadas generalizadas $\longrightarrow q_1 = \theta_1$, $q_2 = \theta_2$
(2 g. l.)

$$\begin{cases} x_{CM_1} = a\sin\theta_1 + \frac{L}{2} \\ y_{CM_1} = -a\cos\theta_1 \end{cases} \longrightarrow \begin{cases} \dot{x}_{CM_1} = a\dot{\theta}_1\cos\theta_1 \\ \dot{y}_{CM_1} = a\dot{\theta}_1\sin\theta_1 \end{cases}$$

$$\begin{cases} x_{CM_2} = a\sin\theta_1 + a\sin\theta_2 + \frac{L}{2} \\ y_{CM_2} = -a\cos\theta_1 - a\cos\theta_2 \end{cases} \longrightarrow \begin{cases} \dot{x}_{CM_2} = a\dot{\theta}_1\cos\theta_1 + a\dot{\theta}_2\cos\theta_2 \\ \dot{y}_{CM_2} = a\dot{\theta}_1\sin\theta_1 + a\dot{\theta}_2\sin\theta_2 \end{cases}$$

$$T = T_{tras} + \cancelto{0}{T_{rot}}$$

$$T = T_{tras_1} + T_{tras_2} = \frac{1}{2}m_1(\dot{x}_{CM_1}^2 + \dot{y}_{CM_1}^2) + \frac{1}{2}m_2(\dot{x}_{CM_2}^2 + \dot{y}_{CM_2}^2) =$$

$$= \frac{1}{2} m_1 \left(a^2 \dot\theta_1^2 \cos^2\theta_1 + a^2 \dot\theta_1 \sin^2\theta_1 \right) +$$

$$+ \frac{1}{2} m_2 \left(a^2 \dot\theta_1^2 \cos^2\theta_1 + a^2 \dot\theta_2^2 \cos^2\theta_2 + 2a^2 \cos\theta_1 \cos\theta_2 \, \dot\theta_1 \dot\theta_2 + \right.$$

$$\left. + a^2 \dot\theta_1^2 \sin^2\theta_1 + a^2 \dot\theta_2^2 \sin^2\theta_2 + 2a^2 \sin\theta_1 \sin\theta_2 \, \dot\theta_1 \dot\theta_2 \right) =$$

$$= \frac{1}{2} m_1 a^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left(a^2 \dot\theta_1^2 + a^2 \dot\theta_2^2 + 2a^2 \left[\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \right] \dot\theta_1 \dot\theta_2 \right)$$

$$= \frac{1}{2} a^2 (m_1 + m_2) \dot\theta_1^2 + m_2 a^2 \underbrace{\cos(\theta_1 - \theta_2)}_{} \dot\theta_1 \dot\theta_2 + \frac{1}{2} a^2 m_2 \dot\theta_2^2 =$$

pequeñas oscilaciones

$$\cos(\theta_1 - \theta_2) \simeq 1 - \frac{1}{2} (\theta_1 - \theta_2)^2 \simeq 1$$

$$= \frac{1}{2} \underbrace{(m_1 + m_2) a^2}_{m_{11}} \dot\theta_1^2 + \underbrace{m_2 a^2}_{m_{12} = m_{21}} \dot\theta_1 \dot\theta_2 + \frac{1}{2} \underbrace{m_2 a^2}_{m_{22}} \dot\theta_2^2$$

$$\circledast \; T = \frac{1}{2} \sum_{j,k=1}^{2} m_{jk} \dot q_j \dot q_k \longrightarrow \{m\} = \begin{pmatrix} (m_1 + m_2) a^2 & m_2 a^2 \\ m_2 a^2 & m_2 a^2 \end{pmatrix}$$

$V = 0 \Leftrightarrow y = 0$ → Origen de potenciales

$$\left. \begin{aligned} V_1 &= m_1 g \, y_{cm1} \\ V_2 &= m_2 g \, y_{cm2} \end{aligned} \right\} \; V = - m_1 g a \cos\theta_1 - m_2 g a (\cos\theta_1 + \cos\theta_2)$$

$$\circledast \; V = \frac{1}{2} \sum_{j,k=1}^{2} A_{jk} q_j q_k \qquad A_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k}$$

$$\frac{\partial V}{\partial \theta_1} = m_1 g a \sin\theta_1 + m_2 g a \sin\theta_1 = (m_1 + m_2) g a \sin\theta_1$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g a \sin\theta_2$$

$$\frac{\partial^2 V}{\partial \theta_1^2} = (m_1 + m_2)\, g\, a \cos\theta_1 \qquad \frac{\partial^2 V}{\partial \theta_2^2} = m_2\, g\, a \cos\theta_2$$

$$\frac{\partial^2 V}{\partial\theta_1 \partial\theta_2} = \frac{\partial^2 V}{\partial\theta_2 \partial\theta_1} = 0$$

⊛ Pequeñas osc. : $\cos\theta_1 \simeq \cos\theta_2 \simeq 1$

$$A_{11} = (m_1 + m_2)\, g\, a \qquad A_{12} = A_{21} = 0$$

$$A_{22} = m_2\, g\, a$$

$$\boxed{\{A\} = \begin{pmatrix} (m_1+m_2)ga & 0 \\ 0 & m_2 g a \end{pmatrix}}$$

Ec. secular: $\det\left[\{A\} - \omega^2 \{m\}\right] = 0$

$$\det \begin{pmatrix} (m_1+m_2)\left[ga - \omega^2 a^2\right] & -m_2 a^2 \omega^2 \\ -m_2 a^2 \omega^2 & m_2 (ga - a^2\omega^2) \end{pmatrix} = 0$$

$$(m_1+m_2)\, m_2 \, (ga - a^2\omega^2)^2 = m_2^2\, a^4 \omega^4$$

$$(m_1+m_2)\, a^2 (g - a\omega^2)^2 = m_2\, a^4 \omega^4$$

$$(m_1+m_2)\, (g^2 + a^2\omega^4 - 2ga\omega^2) = m_2\, a^2 \omega^4$$

$$m_1 a^2 \omega^4 - 2ga(m_1+m_2)\,\omega^2 + (m_1+m_2)\, g^2 = 0$$

$$\omega^2 = \frac{\cancel{2} g \cancel{a} (m_1+m_2) \pm \sqrt{\cancel{4} g^2 a^2 (m_1+m_2)^2 - \cancel{4}(m_1+m_2) g^2 a^2 m_1}}{\cancel{2}\, m_1 a^2}$$

$$\omega^2 = \frac{g(m_1+m_2) \pm g\sqrt{(m_1+m_2)^2 - m_1(m_1+m_2)}}{m_1 a}$$

$$\omega^2 = \frac{g}{a}\left[\frac{m_1}{m_1} + \frac{m_2}{m_1} \pm \sqrt{\frac{m_1^2 + m_2^2 + 2m_1m_2 - m_1^2 - m_1m_2}{m_1}}\right) =$$

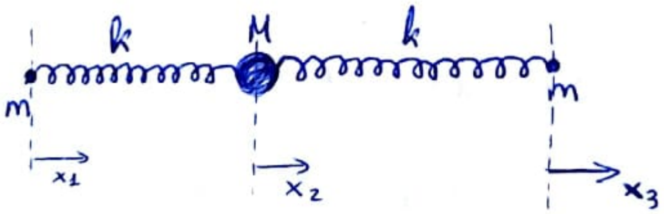
$$= \frac{g}{a}\left[1 + \frac{m_2}{m_1} \pm \sqrt{\frac{m_2(m_2+m_1)}{m_1^2}}\right] =$$

$$= \frac{g}{a}\left[1 + \frac{m_2}{m_1} \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 + \frac{m_2}{m_1}}\right] \Rightarrow$$

$$\omega^2 = \frac{g}{a}\left(1 + \frac{m_2}{m_1} \pm \sqrt{\frac{m_2}{m_1}\left[1 + \frac{m_2}{m_1}\right]}\right)$$

factor no aparece en la sol del c.v.

3.16 Estudiar las vibraciones longitudinales de una molécula triatómica con un átomo central de masa M ligado a dos de masa m mediante resortes de constante k.



3 masas 1D
0 ligaduras } 3 g l.

coordenadas: $\underbrace{x_1, x_2, x_3}$

Desplazamientos respecto a la posic[ión] de equilibrio

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} M \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2$$

$$V = \frac{1}{2} k (x_1 - x_2)^2 + \frac{1}{2} k (x_2 - x_3)^2$$

$$T = \frac{1}{2}\sum_{j,k=1}^{3} m_{jk}\, \dot{x}_j \dot{x}_k \longrightarrow \{m\} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$V = \frac{1}{2}\sum_{j,k=1}^{3} A_{jk}\, x_j x_k$$

$$A_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k}$$

$$\frac{\partial V}{\partial x_1} = k(x_1 - x_2) \quad , \quad \frac{\partial^2 V}{\partial x_1^2} = k \quad , \quad \frac{\partial^2 V}{\partial x_1 \partial x_2} = -k \quad , \quad \frac{\partial^2 V}{\partial x_1 \partial x_3} = 0$$

$$\frac{\partial V}{\partial x_2} = -k(x_1 - x_2) + k(x_2 - x_3) \quad , \quad \frac{\partial^2 V}{\partial x_2^2} = 2k \quad , \quad \frac{\partial^2 V}{\partial x_2 \partial x_3} = -k$$

$$\frac{\partial V}{\partial x_3} = -k(x_2 - x_3) \quad , \quad \frac{\partial^2 V}{\partial x_3^2} = k$$

$$\{A\} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

Ecuación secular: $\det\left[\{A\} - \omega^2\{m\}\right] = 0$

$$\det\begin{pmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{pmatrix} = 0$$

$$(k-\omega^2 m)^2(2k-\omega^2 M) - k^2(k-\omega^2 m) - k^2(k-\omega^2 m) = 0$$

$$(k-\omega^2 m)^2(2k-\omega^2 M) - 2k^2(k-\omega^2 m) = 0$$

$$(k-\omega^2 m)\left[(k-\omega^2 m)(2k-\omega^2 M) - 2k^2\right] = 0$$

(I) $k - \omega^2 m = 0 \quad , \quad \omega^2 = \frac{k}{m} \quad , \quad \boxed{\omega_2 = \sqrt{\frac{k}{m}}}$

(II) $(k-\omega^2 m)(2k-\omega^2 M) - 2k^2 = 0$

$$2k^2 - \omega^2 Mk - 2\omega^2 mk + \omega^4 Mm - 2k^2 = 0$$

$$\omega^4 Mm - \omega^2 k(M+2m) = 0$$

$$\omega^2\left[\omega^2 Mm - k(M+2m)\right] = 0$$

II a $\omega^2 = 0 \longrightarrow \boxed{\omega_1 = 0} \rightsquigarrow$ traslación

II b

$$mM\omega^2 - k(M+2m) = 0$$

$$\omega^2 = \frac{k}{m}\left(1 + \frac{2m}{M}\right) \quad , \quad \boxed{\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}}$$

Coordenadas normales:

$$\eta_1(t) = A_1 \cos(\omega_1 t - \delta_1)$$

$$\eta_2(t) = A_2 \cos(\omega_2 t - \delta_1)$$

$$\eta_3(t) = A_3 \cos(\omega_3 t - \delta_2)$$

¿ Modos normales ?

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\underbrace{\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \text{(autovectores)}}$$

$$\boxed{\left[\{A\} - \omega_n^2 \{m\}\right] \vec{a}_n = 0 \qquad n = 1, 2, 3}$$

$$\vec{a}_1 \rightarrow \vec{a} = (a_1, a_2, a_3)$$

$$\vec{a}_2 \longrightarrow \vec{b} = (b_1, b_2, b_3)$$

$$\vec{a}_3 \longrightarrow \vec{c} = (c_1, c_2, c_3)$$

- $\boxed{n=1}$ $\boxed{\omega_1 = 0} \longrightarrow \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left[\{A\} - \omega_1^2 \{m\}\right] \vec{a} = 0$

$$\left.\begin{array}{l} k(a_1 - a_2) = 0 \Rightarrow \boxed{a_1 = a_2} \\ k(-a_1 + 2a_2 - a_3) = 0 \\ k(a_2 - a_3) = 0 \Rightarrow \boxed{a_2 = a_3} \end{array}\right\} \boxed{a_1 = a_2 = a_3 = a}$$

$$\boxed{\vec{a} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}}$$

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$\bullet\ \boxed{n=2}$ $\quad \boxed{\omega_2 = \sqrt{\frac{k}{m}}}$

$$\left[\{A\} - \omega_2^2 \{M\}\right]\vec{b} = 0 \rightarrow \begin{pmatrix} k-\frac{k}{m}m & -k & 0 \\ -k & 2k-\frac{k}{m}M & -k \\ 0 & -k & k-\frac{k}{m}m \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-k b_2 = 0 \longleftarrow \begin{pmatrix} 0 & -k & 0 \\ -k & k\left(2-\frac{M}{m}\right) & -k \\ 0 & -k & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\hookrightarrow \boxed{b_2 = 0}$

$$k\left(-b_1 + \left(2-\frac{M}{m}\right)b_2 - b_3\right) = 0$$

$$-b_1 - b_3 = 0 \rightarrow \boxed{b_3 = -b_1} \rightarrow \boxed{\vec{b} = \begin{pmatrix} b \\ 0 \\ -b \end{pmatrix}}$$

$\bullet\ \boxed{n=3}$ $\quad \boxed{\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}}$

$$\left[\{A\} - \omega_3^2 \{M\}\right]\vec{c} = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k-\frac{k}{m}m\left(1+\frac{2m}{M}\right) & -k & 0 \\ -k & 2k-M\frac{k}{m}\left(1+\frac{2m}{M}\right) & -k \\ 0 & -k & k-\frac{k}{m}m\left(1+\frac{2m}{M}\right) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-2km}{M} & -k & 0 \\ -k & -\frac{Mk}{m} & -k \\ 0 & -k & \frac{-2km}{M} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-k\left(\frac{2m}{M}c_1 + c_2\right) = 0, \quad \boxed{c_2 = -\frac{2m}{M}c_1}$$

$$-k\left(c_2 + \frac{2m}{M}c_3\right) = 0, \quad \boxed{c_2 = -\frac{2m}{M}c_3}$$

$\Big\}\ \boxed{c_1 = c_3} = c$

$$\boxed{\vec{c} = \begin{pmatrix} c \\ -\frac{2m}{M}c \\ c \end{pmatrix}}$$

Movimiento del sistema $\longrightarrow$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & 0 & -\frac{2m}{M}c \\ a & -b & c \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$

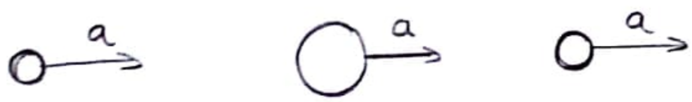
Modos normales

- $n = 1 \longrightarrow \eta_1(t) \neq 0 \qquad \eta_2(t) = \eta_3(t) = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & 0 & -\frac{2m}{M}c \\ a & -b & c \end{pmatrix} \begin{pmatrix} \eta_1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \boxed{\begin{aligned} x_1 &= a\eta_1 \\ x_2 &= a\eta_1 \\ x_3 &= a\eta_1 \end{aligned}}$$

$$\downarrow$$

$$\underbrace{\boxed{\omega_1 = 0} \qquad \boxed{x_1 = x_2 = x_3}}_{\text{Traslación de la molécula:}}$$



- $n = 2 \longrightarrow \eta_2(t) \neq 0 \quad , \quad \eta_1(t) = \eta_3(t) = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & 0 & -\frac{2m}{M}c \\ a & -b & c \end{pmatrix} \begin{pmatrix} 0 \\ \eta_2 \\ 0 \end{pmatrix} \longrightarrow \boxed{\begin{aligned} x_1 &= b\eta_2 \\ x_2 &= 0 \\ x_3 &= -b\eta_2 \end{aligned}}$$

$$\downarrow$$

$$\underbrace{\boxed{\omega_2 = \sqrt{\frac{k}{m}}} \qquad \boxed{\begin{aligned} x_1 &= -x_3 \\ x_2 &= 0 \end{aligned}}}$$

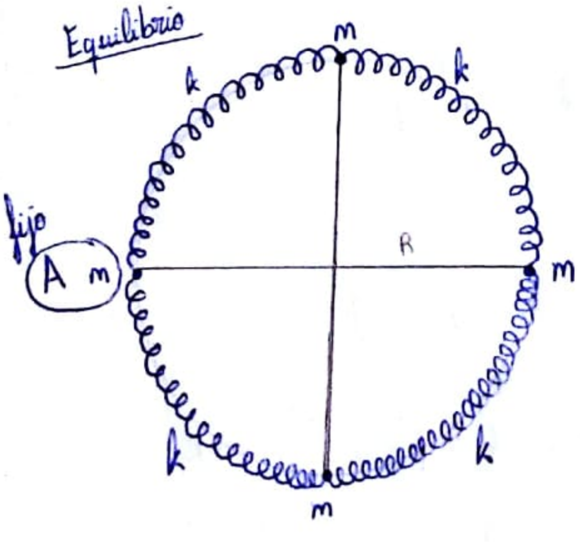
• $\boxed{n=3}$ $\longrightarrow$ $\eta_3(t) \neq 0$, $\eta_1(t) = \eta_2(t) = 0$

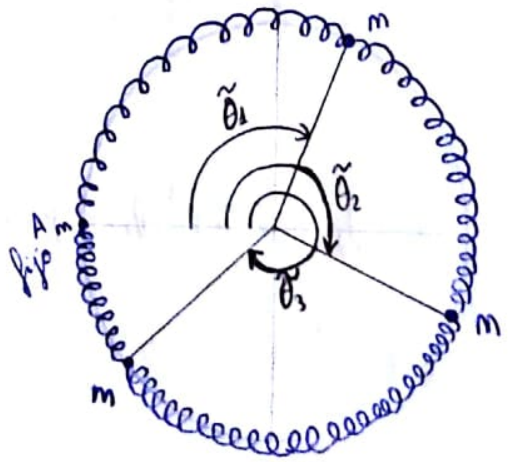
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & 0 & -\frac{2m}{M}c \\ a & -b & c \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \eta_3 \end{pmatrix}$$

$$x_1 = c\,\eta_3$$
$$x_2 = -\frac{2m}{M} c\,\eta_3 \longrightarrow \boxed{x_1 = x_3} \quad \boxed{x_2 = -\frac{2m}{M} x_1} \qquad \omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}$$
$$x_3 = c\,\eta_3$$

3.17 Tres masas iguales (m) $\longrightarrow$ conectadas mediante resortes
↳ se mueven en un círculo.

Punto A fijo ¿Eq. estable? ¿coordenadas normales?
¿frecuencias propias?



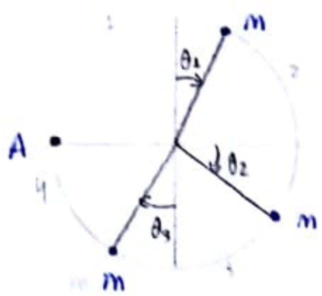


3 partículas $\longrightarrow$ 1D $\longrightarrow$ Coordenadas generalizadas:

$q_1 = \tilde\theta_1$, $q_2 = \tilde\theta_2$, $q_3 = \tilde\theta_3$

Equilibrio $\longrightarrow$ $\tilde\theta_{1_0} = \frac{\pi}{2}$, $\tilde\theta_{2_0} = \pi$, $\tilde\theta_{3_0} = \frac{3\pi}{2}$

Cambio de notación:



$$\theta_1 \equiv \tilde{\theta}_1 - \frac{\pi}{2}$$
$$\theta_2 \equiv \tilde{\theta}_2 - \pi$$
$$\theta_3 \equiv \tilde{\theta}_3 - \frac{3\pi}{2}$$

Desplazamientos respecto a la posición de equilibrio.

$$T = \frac{1}{2} m \left(R^2\dot{\theta}_1^2 + R^2\dot{\theta}_2^2 + R^2\dot{\theta}_3^2 \right)$$

$$V = V_1 + V_2 + V_3 + V_4 = \frac{1}{2} k (R\theta_1)^2 + \frac{1}{2} k (R\theta_2 - R\theta_1)^2 + \frac{1}{2} k (R\theta_3 - R\theta_2)^2 + \frac{1}{2} k (R\theta_3)^2$$

$$\boxed{T = \frac{1}{2} \sum_{j,k=1}^{3} m_{jk} \dot{q}_j \dot{q}_k} \qquad \{m\} = \begin{pmatrix} mR^2 & 0 & 0 \\ 0 & mR^2 & 0 \\ 0 & 0 & mR^2 \end{pmatrix}$$

$$\boxed{V = \frac{1}{2} \sum_{j,k=1}^{3} A_{jk} q_j q_k} \qquad A_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k}$$

$$\frac{\partial V}{\partial \theta_1} = kR^2\theta_1 - kR^2(\theta_2 - \theta_1) = kR^2(2\theta_1 - \theta_2)$$

$$\frac{\partial V}{\partial \theta_2} = kR^2(\theta_2 - \theta_1) - kR^2(\theta_3 - \theta_2) = kR^2(2\theta_2 - \theta_1 - \theta_3)$$

$$\frac{\partial V}{\partial \theta_3} = kR^2(\theta_3 - \theta_2) + kR^2\theta_3 = kR^2(2\theta_3 - \theta_2)$$

$$\frac{\partial^2 V}{\partial \theta_1^2} = 2kR^2 = A_{11} \qquad \frac{\partial^2 V}{\partial \theta_2^2} = 2kR^2 = A_{22}$$

$$\frac{\partial^2 V}{\partial \theta_3^2} = 2kR^2 = A_{33}$$

$$\frac{\partial^2 V}{\partial\theta_1\partial\theta_2} = -kR^2 = A_{12} = A_{21}$$

$$\frac{\partial^2 V}{\partial\theta_1\partial\theta_3} = 0 = A_{13} = A_{31}$$

$$\frac{\partial^2 V}{\partial\theta_2\partial\theta_3} = -kR^2 = A_{23} = A_{32}$$

$$\{A\} = \begin{pmatrix} 2kR^2 & -kR^2 & 0 \\ -kR^2 & 2kR^2 & -kR^2 \\ 0 & -kR^2 & 2kR^2 \end{pmatrix}$$

$$\downarrow$$

$$\{A\} = R^2 \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix}$$

$$\{m\} = R^2 \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$$

Ecuación secular:

$$\det\left[\{A\} - \omega^2\{m\}\right] = 0$$

$$\det\left[R^2 \begin{pmatrix} 2k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 m & -k \\ 0 & -k & 2k-\omega^2 m \end{pmatrix}\right] = 0$$

$$(2k-\omega^2 m)^3 - 2k^2(2k - m\omega^2) = 0$$

$$(2k - m\omega^2)\left[(2k-m\omega^2)^2 - 2k^2\right] = 0$$

I) $2k - m\omega^2 = 0, \quad \omega^2 = \frac{2k}{m}, \quad \omega_2 = \sqrt{\frac{2k}{m}}$

II) $(2k - m\omega^2)^2 - 2k^2 = 0 \longrightarrow$ (*) más fácil:

$$\boxed{\begin{aligned} (2k - m\omega^2)^2 &= 2k^2 \\ 2k - m\omega^2 &= \pm\sqrt{2}\,k \\ \omega^2 &= (2\pm\sqrt{2})\frac{k}{m} \end{aligned}}$$

$$4k^2 + m^2\omega^4 - 4km\omega^2 - 2k^2 = 0$$

$$m^2\omega^4 - 4km\omega^2 + 2k^2 = 0$$

$$\omega_{1,3}^2 = \frac{4km \pm\sqrt{16k^2m^2 - 8k^2m^2}}{2m^2} = \frac{2k\pm\sqrt{2k^2}}{m} = \frac{2k\pm\sqrt{2}\,k}{m}$$

$$\omega_1^2 = (2-\sqrt{2})\frac{k}{m} \rightarrow \omega_1 = \sqrt{(2-\sqrt{2})\frac{k}{m}}$$

$$\omega_3^2 = (2+\sqrt{2})\frac{k}{m} \rightarrow \omega_3 = \sqrt{(2+\sqrt{2})\frac{k}{m}}$$

¿Modos normales? $\longrightarrow$ ¿Vectores propios $\vec{a_n}$ $n=1,2,3$?

$$\vec{a_1} = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \vec{a_2} = \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\vec{a_3} = \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

• $\boxed{n=1}$

$$\omega_1^2 = (2-\sqrt{2})\frac{k}{m} = \left(1-\frac{\sqrt{2}}{2}\right)\frac{2k}{m}$$

$$\left[\{A\} - \omega_1^2\{m\}\right]\vec{a} = 0$$

$$R^2 \begin{pmatrix} 2k - \left(1-\frac{\sqrt{2}}{2}\right)\frac{2k}{m}m & -k & 0 \\ -k & 2k - \left(1-\frac{\sqrt{2}}{2}\right)\frac{2k}{m}m & -k \\ 0 & -k & 2k - \left(1-\frac{\sqrt{2}}{2}\right)\frac{2k}{m}m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$kR^2 \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sqrt{2}\, a_1 - a_2 = 0 \ , \quad \boxed{a_2 = \sqrt{2}\, a_1}$$

$$-a_1 + \sqrt{2}\, a_2 - a_3 = 0 \ , \quad -a_1 + 2a_1 - a_3 = 0$$

$$\boxed{a_1 = a_3}$$

$$\boxed{\vec{a} = \begin{pmatrix} a \\ \sqrt{2}\, a \\ a \end{pmatrix}}$$

• $\boxed{n=2}$

$$\omega_2^2 = \frac{2k}{m}$$

$$R^2 \begin{pmatrix} 2k - \frac{2k}{m}m & -k & 0 \\ -k & 2k - \frac{2k}{m}m & -k \\ 0 & -k & 2k - \frac{2k}{m}m \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\{A\} - \omega_2^2\{m\}\right]\vec{b} = 0 \qquad kR^2 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\vec{b} = \begin{pmatrix} b \\ 0 \\ -b \end{pmatrix}} \longleftarrow \boxed{b_2 = 0} \qquad -b_1 - b_3 = 0 \ , \quad \boxed{b_3 = -b_1}$$

• $\boxed{n=3}$

$$\omega_3^2 = (2+\sqrt{2})\frac{k}{m} = \left(1+\frac{\sqrt{2}}{2}\right)\frac{2k}{m}$$

$$\left[\{A\} - \omega_3^2\{m\}\right]\vec{C} = 0$$

$$R^2\begin{pmatrix} 2k-\left(1+\frac{\sqrt{2}}{2}\right)\frac{2k}{\cancel{m}}\cancel{m} & -k & 0 \\ -k & 2k-\left(1+\frac{\sqrt{2}}{2}\right)\frac{2k}{m}m & -k \\ 0 & -k & 2k-\left(1+\frac{\sqrt{2}}{2}\right)\frac{2k}{m}m \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$kR^2\begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

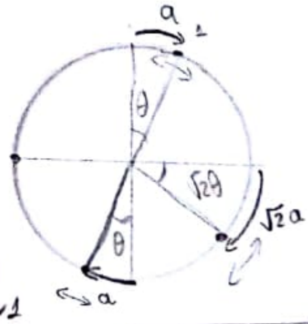
$$\left.\begin{aligned} -\sqrt{2}\,C_1 - C_2 = 0 &\Longrightarrow \boxed{C_2 = -\sqrt{2}\,C_1} \\ -C_2 - \sqrt{2}\,C_3 = 0 &\Longrightarrow C_2 = -\sqrt{2}\,C_3 \end{aligned}\right\} \boxed{C_1 = C_3}$$

$$\boxed{\vec{C} = \begin{pmatrix} c \\ -\sqrt{2}c \\ c \end{pmatrix}}$$

→ coordenadas normales

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ \sqrt{2}a & 0 & -\sqrt{2}c \\ a & -b & c \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

→ <u>Modo 1</u> $\quad \eta_2(t) = \eta_3(t) = 0 \;, \quad \eta_1(t) \neq 0$



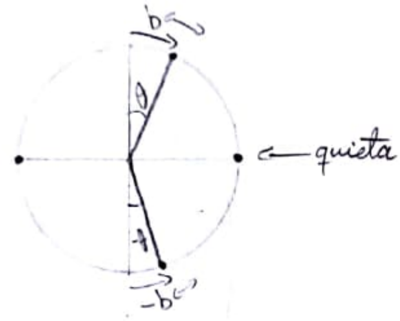
$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ \sqrt{2}a & 0 & -\sqrt{2}c \\ a & -b & c \end{pmatrix}\begin{pmatrix} \eta_1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{aligned} \theta_1 &= a\,\eta_1 \\ \theta_2 &= \sqrt{2}\,a\,\eta_1 \\ \theta_3 &= a\,\eta_1 \end{aligned}$$

$$\boxed{\omega_1^2 = (2-\sqrt{2})\frac{k}{m}}$$

→ <u>Modo 2</u> $\eta_1(t) = \eta_3(t) = 0$, $\eta_2(t) \neq 0$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ \sqrt{2}a & 0 & -\sqrt{2}c \\ a & -b & c \end{pmatrix} \begin{pmatrix} 0 \\ \eta_2 \\ 0 \end{pmatrix} \rightarrow \begin{aligned} \theta_1 &= b\eta_2 \\ \theta_2 &= 0 \\ \theta_3 &= -b\eta_2 \end{aligned}$$

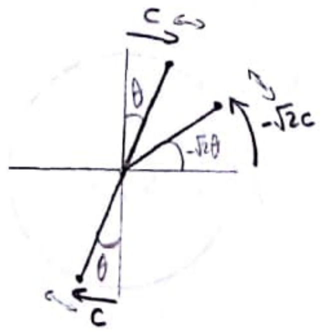
$$\boxed{\omega_2^2 = \frac{2k}{m}}$$



→ <u>Modo 3</u> $\eta_1(t) = \eta_2(t) = 0$, $\eta_3(t) \neq 0$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ \sqrt{2}a & 0 & -\sqrt{2}c \\ a & -b & c \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \eta_3 \end{pmatrix} \rightarrow \begin{aligned} \theta_1 &= c\eta_3 \\ \theta_2 &= -\sqrt{2}c\eta_3 \\ \theta_3 &= c\eta_3 \end{aligned}$$

$$\boxed{\omega_3^2 = (2+\sqrt{2})\frac{k}{m}}$$



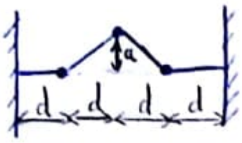
3 18

3Ejs

Cuerda discreta $\longrightarrow$ 3 partículas

$t=0 \longrightarrow$ partícula central desplazada $a \longrightarrow$ se deja libre desde el reposo.

¿ movimiento del sistema ?



$t=0$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2\cdot 4}\right) \qquad n=1,2,3$$
$$\omega_0 = \sqrt{\frac{Z}{md}}$$

$$\omega_1 = 2\sqrt{\frac{Z}{md}}\sin\left(\frac{\pi}{8}\right) \qquad \omega_2 = 2\sqrt{\frac{Z}{md}}\sin\left(\frac{\pi}{4}\right) = \sqrt{\frac{2Z}{md}}$$

$$\omega_3 = 2\sqrt{\frac{Z}{md}}\sin\left(\frac{3\pi}{8}\right)$$

$$q_j(t) = \sum_{n=1}^{n} a_n \sin\left(j\frac{n\pi}{3+1}\right)\cos(\omega_n t - \theta_n)$$

$$q_1(t) = a_1 \sin\left(\frac{\pi}{4}\right)\cos(\omega_1 t - \theta_1) + a_2\sin\left(\frac{\pi}{2}\right)\cos(\omega_2 t - \theta_2) + a_3\sin\left(\frac{3\pi}{4}\right)\cos(\omega_3 t - \theta_3)$$
$$q_2(t) = a_1 \sin\left(\frac{\pi}{2}\right)\cos(\omega_1 t - \theta_1) + a_2\sin(\pi)\cos(\omega_2 t - \theta_2) + a_3\sin\left(\frac{3\pi}{2}\right)\cos(\omega_3 t - \theta_3)$$
$$q_3(t) = a_1 \sin\left(\frac{3\pi}{4}\right)\cos(\omega_1 t - \theta_1) + a_2\sin\left(\frac{3\pi}{2}\right)\cos(\omega_2 t - \theta_2) + a_3\sin\left(\frac{9\pi}{4}\right)\cos(\omega_3 t - \theta_3)$$

c.i.
$$\dot q_j(0) = 0 \Rightarrow \theta_1 = \theta_2 = \theta_3 = 0 \quad (\text{de forma que } \sin(\omega_n t - \theta_n) \text{ se anula en } t=0)$$
$$q_1(0) = q_3(0) = 0$$
$$q_2(0) = a$$

$$q_1(0) = \frac{\sqrt2}{2}a_1 + a_2 + \frac{\sqrt2}{2}a_3 = 0 \longrightarrow \boxed{a_2 = 0}$$
$$q_2(0) = a_1 - a_3 = a$$
$$q_3(0) = a_1\frac{\sqrt2}{2} - a_2 + \frac{\sqrt2}{2}a_3 = 0$$

$$\sqrt2 a_1 + \sqrt2 a_3 = 0$$
$$a_1 = -a_3$$
$$2a_1 = a \;.\; \boxed{a_1 = \frac{a}{2} = -a_3}$$

$$q_1(t) = \frac{\sqrt{2}}{4} a \left(\cos(\omega_1 t) - \cos(\omega_3 t)\right)$$

$$\omega_1 = 2\sqrt{\frac{Z}{md}} \sin\left(\frac{\pi}{8}\right)$$

$$q_2(t) = \frac{a}{2} \left(\cos(\omega_1 t) - \cos(\omega_3 t)\right)$$

$$\omega_3 = 2\sqrt{\frac{Z}{md}} \sin\left(\frac{3\pi}{8}\right)$$

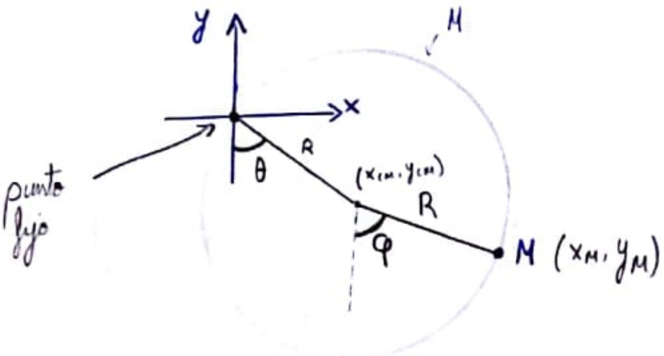
$$q_3(t) = \frac{\sqrt{2}}{4} a \left(\cos(\omega_1 t) - \cos(\omega_3 t)\right)$$

3.19 Aro delgado $\to R$, $\to M$. Oscila en su plano en torno a uno de sus puntos fijo.

masa $M \longrightarrow$ se mueve sin rozamiento sobre el aro

Oscilaciones pequeñas $\quad \boxed{I = 2MR^2}$

¿frecuencias propias? ¿modos normales?



$$\begin{cases} x_{CM} = R\sin\theta \\ y_{CM} = -R\cos\theta \end{cases}$$

$$\begin{cases} x_M = R\sin\theta + R\sin\varphi \\ y_M = -R\cos\theta - R\cos\varphi \end{cases}$$

Coordenadas gen $\longrightarrow \theta, \varphi$

$$T = T_{aro} + T_M$$

$$T_{aro} = T_{rot} = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} 2MR^2\dot{\theta}^2 = MR^2\dot{\theta}^2$$

$$T_M = \frac{1}{2} M (\dot{x}_M^2 + \dot{y}_M^2) = \frac{1}{2} MR^2 \left[\dot{\theta}^2\cos^2\theta + \dot{\varphi}^2\cos^2\varphi + 2\dot{\theta}\dot{\varphi}\cos\theta\cos\varphi + \right.$$

$$\left. + \dot{\theta}^2\sin^2\theta + \dot{\varphi}^2\sin^2\varphi + 2\dot{\theta}\dot{\varphi}\sin\theta\sin\varphi\right] =$$

$$\begin{cases} \dot{x}_M = R\dot{\theta}\cos\theta + R\dot{\varphi}\cos\varphi \\ \dot{y}_M = R\dot{\theta}\sin\theta + R\dot{\varphi}\sin\varphi \end{cases}$$

$$= \frac{1}{2} M R^2 \left[\dot{\theta}^2 + \dot{\varphi}^2 + 2\cos(\theta-\varphi)\,\dot{\theta}\dot{\varphi}\right]$$

$$T = \frac{1}{2} MR^2 \left[\dot{\theta}^2 + \dot{\varphi}^2 + 2\cos(\theta-\varphi)\,\dot{\theta}\dot{\varphi}\right] + MR^2\dot{\theta}^2 =$$

$$= \frac{1}{2}\underbrace{3MR^2}_{m_{11}}\dot{\theta}^2 + \frac{1}{2}\underbrace{MR^2}_{m_{22}}\dot{\varphi}^2 + \underbrace{MR^2\cos(\theta-\varphi)}_{m_{12}=m_{21}}\dot{\theta}\dot{\varphi}$$

$$\boxed{T = \frac{1}{2}\sum_{j,k=1}^{2} m_{j,k}\,\dot{q}_j\dot{q}_k}$$

Oscilaciones pequeñas en torno al equilibrio $(\theta=0,\ \varphi=0)$

$$\cos(\theta-\varphi) \simeq 1 - \frac{1}{2}\underbrace{(\theta-\varphi)^2}_{\simeq 0} \simeq 1$$

$$T \simeq \frac{1}{2}\,3MR^2\,\dot{\theta}^2 + \frac{1}{2}MR^2\dot{\varphi}^2 + MR^2\dot{\theta}\dot{\varphi}$$

$$\{m\} = \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix} = MR^2\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

$V = 0 \leftrightarrow y = 0 \longrightarrow$ Origen de potenciales

$$V = Mg\,y_{CM} + Mg\,y_M = -MgR\cos\theta - MgR(\cos\theta + \cos\varphi) =$$

$$= -MgR\left[2\cos\theta + \cos\varphi\right]$$

Aprox. oscilaciones pequeñas $(\theta \simeq 0,\ \varphi \simeq 0)$

$$\boxed{V = \frac{1}{2}\sum_{j,k=1}^{2} A_{jk}\,q_j q_k}$$

$$V \simeq -MgR\left[2\left(1-\frac{1}{2}\theta^2\right) + \left(1-\frac{1}{2}\varphi^2\right)\right] =$$

$$= -3MgR + MgR\theta^2 + \frac{1}{2}MgR\varphi^2$$

$$A_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k} \qquad \frac{\partial V}{\partial \theta} = 2MgR\theta \qquad A_{11} = \frac{\partial^2 V}{\partial \theta^2} = 2MgR$$

$$\frac{\partial V}{\partial \varphi} = MgR\varphi \qquad A_{12} = A_{21} = \frac{\partial^2 V}{\partial \theta \partial \varphi} = 0$$

$$A_{22} = \frac{\partial^2 V}{\partial \varphi^2} = MgR$$

$$\{A\} = \begin{pmatrix} 2MgR & 0 \\ 0 & MgR \end{pmatrix}$$

<u>Ecuación secular</u>: $\det\left[\{A\} - \omega^2\{m\}\right] = 0$

$$\det\begin{pmatrix} 2MgR - 3MR^2\omega^2 & -MR^2\omega^2 \\ -MR^2\omega^2 & MgR - MR^2\omega^2 \end{pmatrix} = 0$$

⊛ Unidades naturales $\longrightarrow$ $\boxed{M = g = R = 1}$

$$\det\begin{pmatrix} 2-3\omega^2 & -\omega^2 \\ -\omega^2 & 1-\omega^2 \end{pmatrix} = 0$$

$\lambda = \omega^2$

$$\det\begin{pmatrix} 2-3\lambda & -\lambda \\ -\lambda & 1-\lambda \end{pmatrix} = 0, \qquad (2-3\lambda)(1-\lambda) - \lambda^2 = 0$$

$$2 - 2\lambda - 3\lambda + 3\lambda^2 - \lambda^2 = 0$$

$$2\lambda^2 - 5\lambda + 2 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-16}}{2\cdot 2} = \frac{5\pm 3}{4} = \begin{cases} = 2 \\ = \frac{1}{2} \end{cases}$$

$\omega_1^2 = \frac{1}{2}$

$\omega_1 = \sqrt{2}/2$

$\omega_2^2 = 2$

$\omega_2 = \sqrt{2}$

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{g}{2R}} \\ \omega_2 &= \sqrt{\frac{2g}{R}} \end{aligned}}$$

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Vectores propios $\quad \left[\{A\} - \omega_n^2 \{m\}\right] \vec{a}_n = 0$

• $\boxed{n=1}$ $\quad \vec{a}_1 = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \omega_1^2 = \dfrac{g}{2R}$

$$\begin{pmatrix} 2MgR - \frac{g}{2\cancel{R}} 3MR^{\cancel{2}} & -MR^{\cancel{2}} \frac{g}{2\cancel{R}} \\ -MR^{\cancel{2}} \frac{g}{2\cancel{R}} & MgR - MR^{\cancel{2}} \frac{g}{2\cancel{R}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$MgR \begin{pmatrix} 2 - \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 - \frac{1}{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{2} a_1 - \frac{1}{2} a_2 = 0 \,, \quad \boxed{a_1 = a_2 \equiv a}$$

$$\boxed{\vec{a} = \begin{pmatrix} a \\ a \end{pmatrix}}$$

• $\boxed{n=2}$ $\quad \vec{a}_2 = \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \omega_2^2 = \dfrac{2g}{R}$

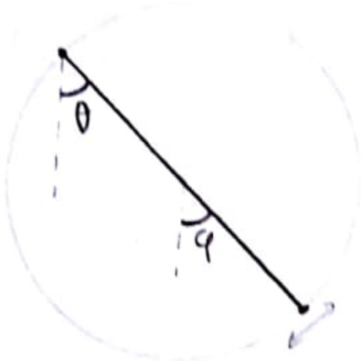
$$\begin{pmatrix} 2MgR - \frac{2g}{\cancel{R}} 3MR^{\cancel{2}} & -MR^{\cancel{2}} \frac{2g}{\cancel{R}} \\ -MR^{\cancel{2}} \frac{2g}{\cancel{R}} & MgR - \frac{2g}{\cancel{R}} MR^{2} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boxed{\vec{b} = \begin{pmatrix} b \\ -2b \end{pmatrix}}$$

$$MgR \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{array}{c} -2b_1 - b_2 = 0 \\ \boxed{b_2 = -2b_1} \end{array}$$

Modos normales:

• **Modo 1** $\qquad \omega_1^2 = \dfrac{g}{2R} \qquad \eta_2(t) = 0 \ , \ \eta_1(t) \neq 0$

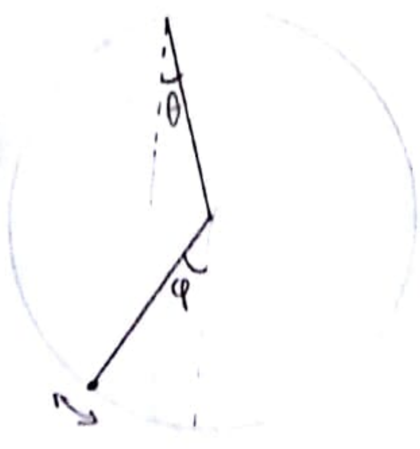
$$\begin{pmatrix} a & b \\ a & -2b \end{pmatrix} \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \longrightarrow \left.\begin{aligned} \theta &= a\,\eta_1 \\ \varphi &= a\,\eta_1 \end{aligned}\right\} \Rightarrow \boxed{\theta = \varphi} \ \forall t$$



(oscilaciones pequeñas)

• **Modo 2** $\qquad \omega_2^2 = \dfrac{2g}{R} \qquad \eta_1(t) = 0 \ , \ \eta_2(t) \neq 0$

$$\begin{pmatrix} a & b \\ a & -2b \end{pmatrix} \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \longrightarrow \left.\begin{aligned} \theta &= b\,\eta_2 \\ \varphi &= -2b\,\eta_2 \end{aligned}\right\} \ \boxed{\varphi = -2\theta}$$



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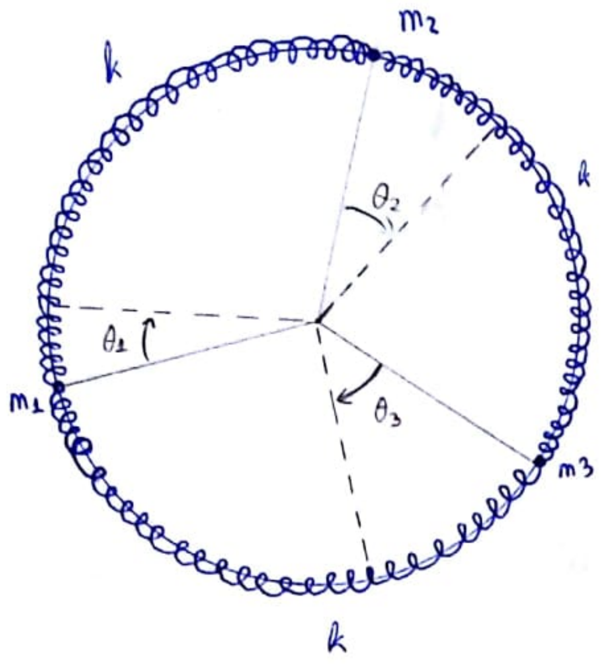


→ Coordenadas generalizadas: $q_1 = \theta_1$, $q_2 = \theta_2$, $q_3 = \theta_3$

$$T = \frac{1}{2} m_1 (R\dot{\theta}_1)^2 + \frac{1}{2} m_2 (R\dot{\theta}_2)^2 + \frac{1}{2} m_3 (R\dot{\theta}_3)^2 =$$

$$= \frac{1}{2} R^2 \left[m_1 \dot{\theta}_1^2 + m_2 \dot{\theta}_2^2 + m_3 \dot{\theta}_3^2 \right]$$

$$V = \frac{1}{2} k [R(\theta_1 - \theta_3)]^2 + \frac{1}{2} k [R(\theta_2 - \theta_1)]^2 + \frac{1}{2} k [R(\theta_3 - \theta_2)]^2 =$$

$$= \frac{1}{2} k R^2 \left[(\theta_1 - \theta_3)^2 + (\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 \right] =$$

$$= \frac{1}{2} k R^2 \left[2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3 - 2\theta_1\theta_3 \right] =$$

$$= k R^2 \left[\theta_1^2 + \theta_2^2 + \theta_3^2 - \theta_1\theta_2 - \theta_2\theta_3 - \theta_1\theta_3 \right]$$

$$T = \frac{1}{2} \sum_{j,k=1}^{3} m_{jk} \dot{q}_j \dot{q}_k \qquad \{m\} = \begin{pmatrix} m_1 R^2 & 0 & 0 \\ 0 & m_2 R^2 & 0 \\ 0 & 0 & m_3 R^2 \end{pmatrix} = R^2 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

$$V = \frac{1}{2} \sum_{j,k=1}^{3} A_{jk} q_j q_k \qquad A_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k}$$

$$V = kR^2\left[\theta_1^2+\theta_2^2+\theta_3^2-\theta_1\theta_2-\theta_2\theta_3-\theta_1\theta_3\right]$$

$$\frac{\partial V}{\partial \theta_1} = kR^2\left[2\theta_1-\theta_2-\theta_3\right]$$

$$\frac{\partial^2 V}{\partial \theta_1^2} = 2kR^2 \qquad \frac{\partial^2 V}{\partial\theta_1\partial\theta_2} = -kR^2 \qquad \frac{\partial^2 V}{\partial\theta_1\partial\theta_3} = -kR^2$$

$$\frac{\partial V}{\partial \theta_2} = kR^2\left[2\theta_2-\theta_1-\theta_3\right]$$

$$\frac{\partial^2 V}{\partial \theta_2^2} = 2kR^2 \qquad \frac{\partial^2 V}{\partial\theta_2\partial\theta_3} = -kR^2$$

$$\frac{\partial V}{\partial \theta_3} = kR^2\left[2\theta_3-\theta_1-\theta_2\right] \quad , \quad \frac{\partial^2 V}{\partial \theta_3^2} = 2kR^2$$

$$\{A\} = kR^2\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Ecuación secular: $\det\left[\{A\}-\omega^2\{m\}\right]=0$

$$\det\begin{pmatrix} 2k-\omega^2 m_1 & -k & -k \\ -k & 2k-\omega^2 m_2 & -k \\ -k & -k & 2k-\omega^2 m_3 \end{pmatrix} = 0$$

⊛ Suponiendo $m_1=m_2=m_3=m$:

$$(2k-\omega^2 m)^3 - 2k^3 - 3k^2(2k-\omega^2 m) = 0$$

$$(2k-\omega^2 m)^3 - 8k^3 + 3k^2\omega^2 m = 0$$

$$8k^3 - 12k^2\omega^2 m + 6k\omega^4 m^2 - \omega^6 m^3 - 8k^3 + 3k^2\omega^2 m = 0$$

$$\omega^2\left(-\omega^4 m^3 + 6k\omega^2 m^2 - 9k^2 m\right) = 0$$

$\rightarrow \boxed{\omega_1 = 0}$

$\rightarrow \quad -m^3\omega^4 + 6km^2\omega^2 - 9k^2m = 0$

$$\omega^2 = \frac{-6km^2 \pm \sqrt{36k^2m^4 - 36k^2m^4}}{-2m^3} = 3\frac{k}{m}$$

$\hookrightarrow \boxed{\omega_2 = \sqrt{\frac{3k}{m}} = \omega_3}$

∘ Vectores propios: $\left[\{A\} - \omega_n^2\{m\}\right]\vec{a_n} = 0$

· $\boxed{n=1}$ $\quad \vec{a_1} = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \omega_1 = 0$

$$kR^2\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2a_1 - a_2 - a_3 = 0$

$a_3 = 2a_1 - a_2 \longrightarrow \boxed{a_3 = a}$

$-a_1 + 2a_2 - a_3 = 0$

$-a_1 + 2a_2 - 2a_1 + a_2 = 0$

$3(a_2 - a_1) = 0 \Rightarrow \boxed{a_2 = a_1} = a$

$$\vec{a} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

· $\boxed{n=2}$ $\quad \vec{a_2} = \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \omega_2 = \sqrt{\frac{3k}{m}}$

$$kR^2\begin{pmatrix} 2-3 & -1 & -1 \\ -1 & 2-3 & -1 \\ -1 & -1 & 2-3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ -b_1 - b_2 \end{pmatrix}$$

$-b_1 - b_2 - b_3 = 0 \longrightarrow b_3 = -b_1 - b_2$

• $\boxed{n=3}$ $\quad \vec{a}_3 = \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \omega_3 = \sqrt{\frac{3k}{m}}$

$$kR^2 \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix}$$

$$-c_1 - c_2 - c_3 = 0 \longrightarrow c_3 = -c_1 - c_2$$

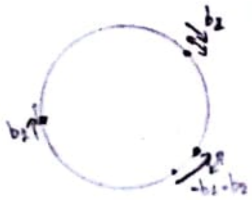
• Modos normales

• 1 $\quad \omega_1 = 0 \quad \eta_1(t) \neq 0 \ , \ \eta_2(t) = \eta_3(t) = 0$

$$\begin{pmatrix} a & b_1 & c_1 \\ a & b_2 & c_2 \\ a & -b_1-b_2 & -c_1-c_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \rightarrow \left.\begin{aligned} \theta_1 &= a\eta_1 \\ \theta_2 &= a\eta_1 \\ \theta_3 &= a\eta_1 \end{aligned}\right\} \boxed{\theta_1 = \theta_2 = \theta_3}$$

• 2 $\quad \omega_2 = \sqrt{\frac{3k}{m}} \quad \eta_2(t) \neq 0 \ , \ \eta_1(t) = \eta_3(t) = 0$

$$\begin{pmatrix} a & b_1 & c_1 \\ a & b_2 & c_2 \\ a & -b_1-b_2 & -c_1-c_2 \end{pmatrix} \begin{pmatrix} 0 \\ \eta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \longrightarrow \left.\begin{aligned} \theta_1 &= b_1 \eta_2 \\ \theta_2 &= b_2 \eta_2 \\ \theta_3 &= (-b_1 - b_2)\eta_2 \end{aligned}\right\} \theta_3 = -(\theta_1 + \theta_2)$$

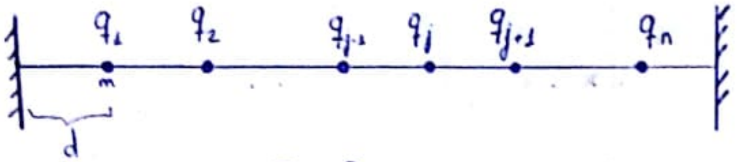


3. $\omega_3 = \sqrt{\frac{3k}{m}}$ $\quad \eta_3(t) \neq 0, \quad \eta_1(t) = \eta_2(t) = 0$

$$\begin{pmatrix} a & b_1 & c_1 \\ a & b_2 & c_2 \\ a & -b_1-b_2 & -c_1-c_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \longrightarrow \left. \begin{aligned} \theta_1 &= c_1 \eta_3 \\ \theta_2 &= c_2 \eta_3 \\ \theta_3 &= (-c_1-c_2)\eta_3 \end{aligned} \right\} \theta_3 = -(\theta_1+\theta_2)$$

Tema 4 : Ondas

→ La cuerda continua como límite de la discreta



(1) $m\ddot{q}_j = \frac{z}{d}\left[(q_{j+1}-q_j)-(q_j-q_{j-1})\right]$ ⊛ $m\omega_0^2 = \frac{z}{d}$

Paso de discreta a continua:

$\left.\begin{array}{l}\cdot\ n\to\infty\\ \cdot\ m\to 0\\ \cdot\ d\to 0\end{array}\right\}$ de tal forma que: $\left\{\begin{array}{l}\frac{m}{d} = \rho \equiv \text{densidad lineal} = \text{cte}\\ (n+1)d = L \equiv \text{longitud} = \text{cte}\end{array}\right.$

lím. al continuo ⟶ $\boxed{x \equiv jd}$

$q_j(t) \equiv q(jd,t) \longrightarrow q(x,t)$

$q_{j-1}(t) \equiv q((j-1)d,t) \longrightarrow q(x-d,t)$

$q_{j+1}(t) \equiv q((j+1)d,t) \longrightarrow q(x+d,t)$

(1) Dividiendo ambos miembros entre d:

$$\frac{m}{d}\ddot{q}_j = m\omega_0^2\left[\frac{q_{j+1}-q_j}{d} - \frac{q_j-q_{j-1}}{d}\right]$$

$\lim_{d\to 0}$

$$\rho\,\ddot{q}_j = m\omega_0^2 \lim_{d\to 0}\left[\frac{q(x+d,t)-q(x,t)}{d} - \frac{q(x,t)-q(x-d,t)}{d}\right]$$

$$\rho \ddot{q}_j = m\omega_0^2 \left[\lim_{d\to 0} \frac{q(x+d,t) - q(x,t)}{d} - \lim_{d\to 0} \frac{q(x,t) - q(x-d,t)}{d} \right]$$

$$\lim_{d\to 0} \frac{q(x+d,t) - q(x,t)}{d} = \left.\frac{\partial q}{\partial x}\right|_{x=x=jd}$$

$$\lim_{d\to 0} \frac{q(x,t) - q(x-d,t)}{d} = \left.\frac{\partial q}{\partial x}\right|_{x=x-d=(j-1)d}$$

$$\rho \ddot{q}_j(x,t) = \frac{Z}{d}\left[\left.\frac{\partial q}{\partial x}\right|_x - \left.\frac{\partial q}{\partial x}\right|_{x-d} \right]$$

$$\rho \ddot{q}_j(x,t) = Z\left[\frac{\left.\frac{\partial q}{\partial x}\right|_x - \left.\frac{\partial q}{\partial x}\right|_{x-d}}{d} \right]$$

$\lim_{d\to 0}$

$$\boxed{\rho \frac{d^2 q}{dt^2} = Z\left[\lim_{d\to 0} \frac{\left.\frac{\partial q}{\partial x}\right|_x - \left.\frac{\partial q}{\partial x}\right|_{x-d}}{d} \right] = Z \frac{\partial^2 q}{\partial x^2}}$$

$q(x,t) \longrightarrow x$ no depende de t $\longrightarrow$ $\boxed{\frac{d^2 q}{dt^2} = \frac{\partial^2 q}{\partial t^2}}$

$$\rho \frac{\partial^2 q}{\partial t^2} = Z \frac{\partial^2 q}{\partial x^2}$$

(*) $\left[\frac{Z}{\rho}\right] = \frac{kg\, m s^{-2}}{kg\, m^{-1}} = m^2 s^{-2} = [v^2]$

$\boxed{v \equiv \sqrt{\frac{Z}{\rho}}}$ $\longrightarrow$ velocidad de la onda

$$\boxed{\frac{\partial^2 q}{\partial^2 x} - \frac{1}{Z/\rho} \frac{\partial^2 q}{\partial t^2} = 0} \xrightarrow{(*)} \boxed{\frac{\partial^2 q}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 q}{\partial t^2} = 0}$$

Ecuación de ondas en 1 Dimensión

→ Ecuación de ondas: $\boxed{\dfrac{\partial^2 \varphi}{\partial x^2} - \dfrac{1}{v^2}\dfrac{\partial^2 \varphi}{\partial t^2} = 0}$

⊛ Solución de Euler:

∘ Cambio de variables $\left| \begin{array}{l} \xi \equiv x + vt \\ \eta \equiv x - vt \end{array} \right.$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial \varphi}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial x}}_{=1} = \frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}\right) \cdot \overbrace{\frac{\partial \xi}{\partial x}}^{=1} +$$

$$+ \frac{\partial}{\partial \eta}\left(\frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}\right) \cdot \underbrace{\frac{\partial \eta}{\partial x}}_{=1} = \frac{\partial^2 \varphi}{\partial \xi^2} + 2\frac{\partial^2 \varphi}{\partial \eta \partial \xi} + \frac{\partial^2 \varphi}{\partial \eta^2}$$

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial t}}_{=v} + \frac{\partial \varphi}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial t}}_{=-v} = v\left[\frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \eta}\right]$$

$$\frac{\partial^2 \varphi}{\partial t^2} = v \cdot \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \eta}\right) = v \cdot \left[\frac{\partial}{\partial \xi}\left(\frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \eta}\right) \overbrace{\frac{\partial \xi}{\partial t}}^{=v} +\right.$$

$$\left. + \frac{\partial}{\partial \eta}\left(\frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \eta}\right) \cdot \underbrace{\frac{\partial \eta}{\partial t}}_{=-v}\right] = v^2\left[\frac{\partial^2 \varphi}{\partial \xi^2} - 2\frac{\partial^2 \varphi}{\partial \xi \partial \eta} + \frac{\partial^2 \varphi}{\partial \eta^2}\right]$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial \xi^2} + 2\frac{\partial^2\psi}{\partial\eta\partial\xi} + \frac{\partial^2\psi}{\partial\eta^2}$$

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial \xi^2} - 2\frac{\partial^2\psi}{\partial\eta\partial\xi} + \frac{\partial^2\psi}{\partial\eta^2}$$

Sustituimos en la ecuación de ondas

$$\boxed{\frac{\partial^2\psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0}$$

$$\frac{\partial^2\psi}{\partial \xi^2} + 2\frac{\partial^2\psi}{\partial\eta\partial\xi} + \frac{\partial^2\psi}{\partial\eta^2} - \frac{\partial^2\psi}{\partial \xi^2} + 2\frac{\partial^2\psi}{\partial\eta\partial\xi} - \frac{\partial^2\psi}{\partial\eta^2} = 0$$

$$\boxed{\frac{\partial^2\psi}{\partial\eta\partial\xi} = 0}$$

$$\psi(\eta,\xi) = f(\xi) + g(\eta)$$

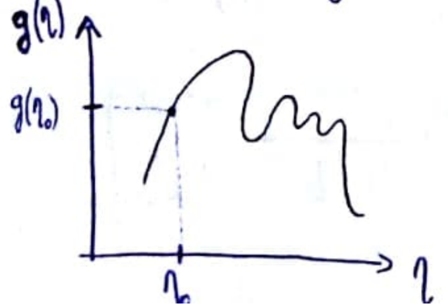
Deshacemos el cambio de variables:

$$\boxed{\psi(x,t) = f(x+vt) + g(x-vt)}$$

Solución de la ecuación de ondas

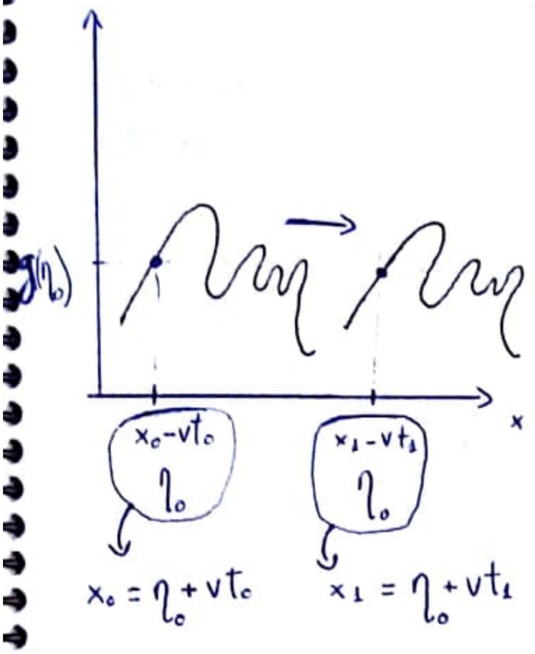
→ Interpretación física:

Sea $g(x-vt) = g(\eta)$



Sea $g(\eta_0) = cte$ el valor que toma $g(\eta)$ evaluada en $\eta_0 = cte = x - vt$

$$\boxed{x = cte + vt}$$



$g(x - vt)$

↳ representa una función que se propaga con velocidad v hacia la derecha.

⊛ Desarrollo análogo ⟶ $f(x + vt)$

se propaga con velocidad $-v$ hacia la izquierda

⊛ Solución de Bernoulli

⟶ Probemos en la ecuación de ondas una solución:

$$\psi(x,t) = \varphi(x)\, e^{i\omega t}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial x^2} e^{i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \varphi(x)\, e^{i\omega t}$$

$$\frac{\partial^2 \varphi}{\partial x^2} e^{i\omega t} + \frac{\omega^2}{v^2} \varphi(x)\, e^{i\omega t} = 0$$

⊛ $\varphi(x) \Rightarrow \boxed{\dfrac{\partial^2 \varphi}{\partial x^2} = \dfrac{d^2 \varphi}{d x^2}}$

$$\frac{d^2 \varphi}{d x^2} + \frac{\omega^2}{v^2} \varphi(x) = 0$$

⊛ $\boxed{k^2 = \dfrac{\omega^2}{v^2}}$

$$\boxed{\frac{d^2 \varphi}{d x^2} + k^2 \varphi = 0}$$

Ecuación de Helmholtz

$k = \frac{\omega}{v}$ nº de onda

$\omega = 2\pi \nu$

$v = \lambda \nu$

$\lambda = \frac{2\pi}{k}$ longitud de onda

Solución general de la ecuación de ondas:

$$\underbrace{\varphi(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\, e^{i\omega_n t}}_{\text{Bernoulli}}$$

* Separación de variables:

$$\varphi(x,t) = \varphi(x)\,\chi(t) \quad \text{(separable)}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial x^2}\,\chi(t) = \chi(t)\,\frac{d^2\varphi}{dx^2}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \varphi(x)\,\frac{\partial^2 \chi}{\partial t^2} = \varphi(x)\,\frac{d^2\chi}{dt^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \varphi}{\partial t^2} = 0$$

$$\chi\,\frac{d^2\varphi}{dx^2} = \frac{\varphi}{v^2}\,\frac{d^2\chi}{dt^2}$$

$$\boxed{\underbrace{\frac{v^2}{\varphi}\,\frac{d^2\varphi}{dx^2}}_{\text{solo depende de } x} = \underbrace{\frac{1}{\chi}\,\frac{d^2\chi}{dt^2}}_{\text{solo depende de } t}}$$

$= cte =$

$$\begin{cases} \dfrac{v^2}{\varphi}\,\dfrac{d^2\varphi}{dx^2} = -\omega^2 = cte \\[2mm] \dfrac{1}{\chi}\,\dfrac{d^2\chi}{dt^2} = -\omega^2 = cte \end{cases}$$

signo − para que las soluciones sean oscilatorias.

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$$\frac{v^2}{\varphi}\frac{d^2\varphi}{dx^2} = -\omega^2 \longrightarrow \frac{d^2\varphi}{dx^2} + \frac{\omega^2}{v^2}\varphi = 0$$

$$\frac{1}{\chi}\frac{d^2\chi}{dt^2} = -\omega^2 \longrightarrow \frac{d^2\chi}{dt^2} + \omega^2\chi = 0$$

Ecuaciones de dos osciladores armónicos

⊛ Def.- $\boxed{k^2 \equiv \frac{\omega^2}{v^2}}$

Soluciones:

$$\begin{cases} \varphi(x) = A e^{ikx} + B e^{-ikx} \\ \chi(t) = C e^{i\omega t} + D e^{-i\omega t} \end{cases}$$

$$\boxed{\Psi(x,t) = \varphi(x)\,\chi(t) \propto e^{\pm ikx} e^{\pm i\omega t}}$$

La solución general de la ecuación de ondas es una combinación lineal de funciones $e^{\pm i(kx \pm \omega t)}$

$A, B, C, D \rightarrow$ c.i.

$\omega \rightarrow$ cte de separación c.c.

Tomemos $\underline{\Psi(x,t) \sim e^{i(kx-\omega t)}}$

$\rightarrow$ Periodicidad espacial: $\boxed{\lambda \equiv \text{longitud de onda}}$

$$\Psi(x+\lambda, t) - \Psi(x,t) = 0$$

$$e^{i[k(x+\lambda)-\omega t]} - e^{i[kx-\omega t]} = 0$$

$$\underbrace{e^{-i\omega t}}_{\neq 0}\underbrace{\left[e^{ik(x+\lambda)} - e^{ikx}\right]}_{= 0} = 0$$

$$k(x+\lambda) - kx = 2n\pi$$

$\boxed{n=1}$ menor periodicidad (con $\lambda \neq 0$)

$k(x+\lambda) - kx = 2\pi$, $k\lambda = 2\pi$, $\boxed{\lambda = \frac{2\pi}{k}}$

$\boxed{k = \frac{2\pi}{\lambda}}$ → número de onda

→ <u>Periodicidad temporal</u>: $\boxed{T \equiv \text{período}}$

$$\Psi(x, t+T) - \Psi(x,t) = 0$$

$$e^{i[kx - \omega(t+T)]} - e^{i[kx - \omega t]} = 0$$

$$kx - \omega(t+T) - \cancel{kx} + \omega t = 2n\pi$$

$$\omega T = 2n\pi$$

$n = 1$ (menor periodicidad con $T \neq 0$)

frecuencia $\boxed{\omega = \frac{2\pi}{T}}$ → $\boxed{T = \frac{2\pi}{\omega}}$

⊛ Como $k^2 = \frac{\omega^2}{v^2}$ ⟶ $v_f = \frac{\omega}{k}$ ⟶ <u>velocidad de fase</u>

$$\boxed{v = \frac{\omega}{k} = \frac{2\pi}{T}\frac{\lambda}{2\pi} = \frac{\lambda}{T}}$$

→ La onda recorre una distancia λ en el tiempo T

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→ Ondas estacionarias

$\underbrace{\text{cuerda fija en sus extremos } \forall t}_{\text{Condición de contorno}}$

$$\varphi(x=0,t) = \varphi(x=L,t) = 0$$

- Sea $\underline{\varphi(x,t) = A\, e^{i(kx-\omega t)}}$

$$\varphi(x=0,t) = A\, e^{-i\omega t} = 0 \quad \forall t \Rightarrow A = 0$$

solución trivial
↳ cuerda quieta

⊗

- Sea $\underline{\varphi(x,t) = A\, e^{i(kx-\omega t)} + B e^{i(-kx-\omega t)}}$

↓ onda que viaja hacia la derecha

↓ onda que viaja hacia la izquierda

$$\varphi(x=0,t) = A\, e^{-i\omega t} + B\, e^{-i\omega t} =$$

$$= (A+B)\, e^{-i\omega t} = 0 \quad \forall t$$

$$A + B = 0 \,, \quad \boxed{A = -B}$$

$$\varphi(x,t) = A\, e^{i(kx-\omega t)} - A\, e^{i(-kx-\omega t)} = A\left[e^{i(kx-\omega t)} - e^{i(-kx-\omega t)}\right] =$$

$$= A e^{-i\omega t}\left[\cos kx + i \sin kx - \cos kx + i \sin kx\right] = A e^{-i\omega t}\, 2i \sin kx =$$

$= 2A\,(\cos\omega t - i \sin\omega t)\, i \sin k$

$$\boxed{\mathrm{Re}\left[\psi(x,t)\right] = 2A \sin kx \sin\omega t} \longrightarrow \text{Onda estacionaria}$$

Condición de contorno: $\psi(x=L,t) = 0 = 2A \sin kL \sin\omega t \quad \forall t$

$A \neq 0$ (solución no trivial)

$\sin(kL) = 0$

$\hookrightarrow kL = (n+1)\pi$

$$\boxed{k_n = (n+1)\frac{\pi}{L} \qquad n = 0,1,2.}$$

sólo existen ondas con estas longitudes de onda $\leftarrow$

$$\boxed{\lambda_n = \frac{2\pi}{k_n} = \frac{2\pi}{(n+1)\frac{\pi}{L}} = \frac{2L}{n+1}}$$

$n = 0,1,2.$

En las ondas estacionarias existen puntos del espacio x_j tales que

$$\psi(x_j,t) = 0 \quad (\forall t) \Longrightarrow \sin k_n x_j = 0 \Rightarrow x_j = \frac{j\pi}{k_n} \qquad j = 0,1,2..$$

$$\boxed{x_j = \frac{1}{2} j \lambda_n, \; j = 0,1,2..}$$

Nodos

p.e.

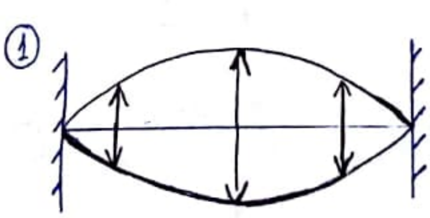
(1) $n=0$

$\lambda_0 = 2L$

$x_j = \frac{1}{2} j\, 2L = jL$

$\underbrace{j=0,\ j=1}$

$\left.\begin{array}{l} x_0 = 0 \\ x_1 = L \end{array}\right\}$ Extremos (únicos nodos)



(2) $n=1$

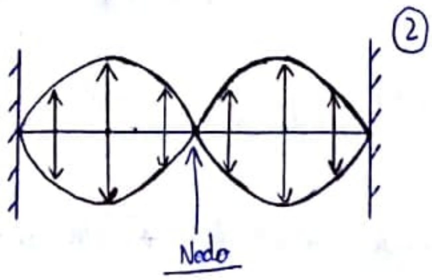
$\lambda_1 = L$

$x_j = \frac{1}{2} j L = \frac{L}{2} j$

$x_0 = 0$

$x_1 = \frac{L}{2} \longrightarrow$ nodo (además de los extremos)

$x_2 = L$



→ Velocidad de fase, dispersión y velocidad de grupo

$$\psi(x,t) = A\, e^{i(kx-\omega t)} \qquad \circledast\ k = \frac{\omega}{v}$$

① Velocidad de fase

$$\boxed{k = \frac{2\pi}{\lambda},\ \omega = \frac{2\pi}{T}}$$

$$\phi = kx - \omega t = cte$$

$$d\phi = k\,dx - \omega\,dt = 0 \longrightarrow \boxed{v_f \equiv \frac{dx}{dt} = \frac{\omega}{k} = \frac{\lambda}{T}}$$

La velocidad de fase es la velocidad que debe tener un observador para que vea siempre la misma fase

② Dispersión

$$\psi(x,t) = \underbrace{\sum_{n=1}^{n} a_n\, e^{i(k_n x - \omega_n t)}}$$

$a_n\, e^{i(k_n x - \omega_n t)} \longrightarrow$ onda monocromática

$$k_n = \frac{2\pi}{\lambda_n}$$

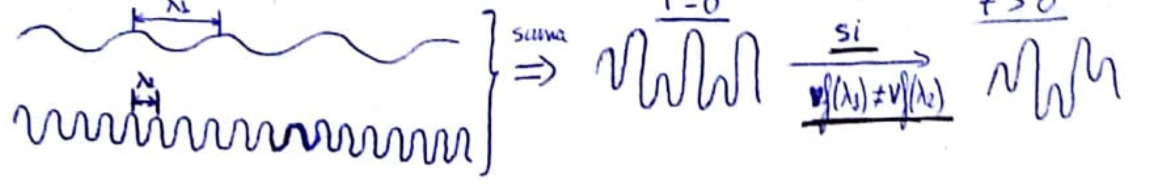
$$v_f = \frac{\omega_n}{k_n}$$

Existe dispersión si existe una dependencia entre la frecuencia ω_n de cada una de estas ondas y su velocidad de fase. → Esto no sucede en el vacío. En general tiene que haber un medio material

↓

Cada onda monocromática se desplaza con una v_f distinta

Pensemos en la superposición de dos ondas que se propagan a distinta v_f:



la onda resultante se deforma

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③ Velocidad de grupo:

Tomamos dos ondas: (de igual amplitud, para que el desarrollo sea más simple)

$$\left.\begin{aligned}\psi_1 &= A\,e^{i(\omega t - kx)}\\ \psi_2 &= A\,e^{i(\Omega t - Kx)}\end{aligned}\right\}\qquad \boxed{\begin{aligned}\Omega &= \omega + \Delta\omega\\ K &= k + \Delta k\end{aligned}}$$

Suponemos también

$$\frac{\Delta\omega}{\omega} \ll 1$$

$$\frac{\Delta k}{k} \ll 1$$

□ Superposición de las dos ondas:

$$\psi(x,t) = \psi_1 + \psi_2 = A\left[e^{i\omega t}e^{-ikx} + e^{i(\omega+\Delta\omega)t}e^{-i(k+\Delta k)x}\right] =$$

$$= A\left[e^{i(\omega+\frac{\Delta\omega}{2})t}\,e^{-i(k+\frac{\Delta k}{2})x}\right]\left[e^{-i\frac{\Delta\omega}{2}t}e^{i\frac{\Delta k}{2}x} + e^{i\frac{\Delta\omega}{2}t}e^{-i\frac{\Delta k}{2}x}\right] =$$

$$= A\left[e^{i(\omega+\frac{\Delta\omega}{2})t}\,e^{-i(k+\frac{\Delta k}{2})x}\right]\underbrace{\left[e^{-i\left(\frac{\Delta\omega t - \Delta k x}{2}\right)} + e^{i\left(\frac{\Delta\omega t - \Delta k x}{2}\right)}\right]}_{e^{i\alpha}+e^{-i\alpha} = 2\cos\alpha} =$$

$$= A\left[e^{i(\omega+\frac{\Delta\omega}{2})t}\,e^{-i(k+\frac{\Delta k}{2})x}\right]2\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right) =$$

$$= 2A\left\{\left[\cos(\omega+\tfrac{\Delta\omega}{2})t + i\sin(\omega+\tfrac{\Delta\omega}{2})t\right]\cdot\left[\cos(k+\tfrac{\Delta k}{2})x - i\sin(k+\tfrac{\Delta k}{2})x\right]\right\}\cdot\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$$

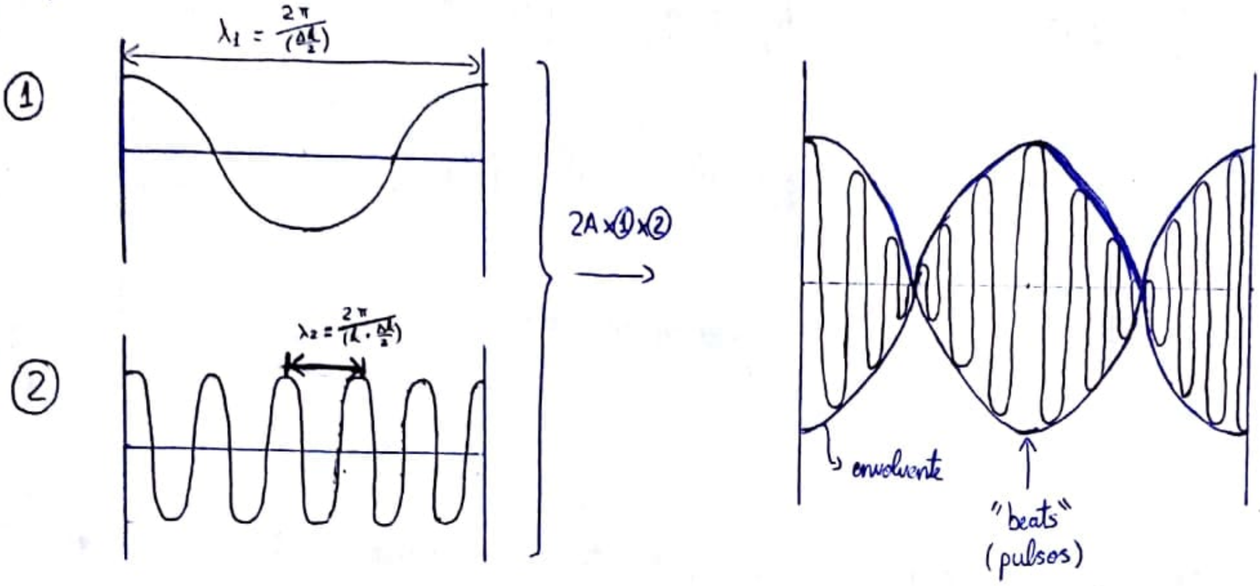
Tomamos la parte real

$$\mathrm{Re}\left[\psi(x,t)\right] = 2A\left[\cos\left[(\omega+\tfrac{\Delta\omega}{2})t\right]\cdot\cos\left[(k+\tfrac{\Delta k}{2})x\right] + \sin\left[(\omega+\tfrac{\Delta\omega}{2})t\right]\sin\left[(k+\tfrac{\Delta k}{2})x\right]\right]$$

$$\cdot\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right) = 2A\,\underbrace{\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)}_{①}\,\underbrace{\cos\left[(\omega+\tfrac{\Delta\omega}{2})t - (k+\tfrac{\Delta k}{2})x\right]}_{②}$$

Como partimos de $\Delta\omega \ll \omega$ y $\Delta k \ll k$,

podemos decir que ① tiene una frecuencia mucho más baja $\left(\frac{\Delta\omega}{2} \lll \omega + \frac{\Delta\omega}{2}\right)$ y una mayor longitud de onda $\left(\frac{\Delta k}{2} \lll k + \frac{\Delta k}{2}\right)$ que ②



③ Def.- Velocidad de grupo: velocidad de fase de la envolvente (velocidad a la que se propagan los pulsos)

$$\boxed{v_g = v_{f_1} = \frac{\Delta\omega}{\Delta k}} \longrightarrow$$ para un paquete de ondas:

$$v_g = \frac{d\omega}{dk}$$

→ Relación entre v_f y v_g:

$$\lambda = \frac{2\pi}{k}$$

$$v_g = \frac{d\omega}{dk} = \frac{d(k v_f)}{dk} = v_f + k\frac{dv_f}{dk} = v_f + k\,\frac{dv_f}{d\lambda}\,\frac{d\lambda}{dk} =$$

$v_f = \frac{\omega}{k}$, $\omega = k v_f$

$$= v_f - k\,\frac{2\pi}{k^2}\,\frac{dv_f}{d\lambda} = v_f - \frac{2\pi}{k}\,\frac{dv_f}{d\lambda} =$$

$\lambda = \frac{2\pi}{k}$

$$= v_f - \lambda\,\frac{dv_f}{d\lambda}$$

$$\boxed{v_g = v_f - \lambda \frac{dv_f}{d\lambda}}$$

- Casos: Medio

→ Si $\frac{dv_f}{d\lambda} = 0 \longrightarrow$ No dispersivo $\boxed{v_g = v_f}$

→ Si $\frac{dv_f}{d\lambda} > 0 \longrightarrow$ Dispersión normal $\boxed{v_g < v_f}$

→ Si $\frac{dv_f}{d\lambda} < 0 \longrightarrow$ Dispersión anómala $\boxed{v_g > v_f}$

□ A continuación demostraremos que si la relación entre $\omega(k)$ no es lineal, el medio es dispersivo, pues $v_f = \frac{\omega}{k} \neq \boxed{\frac{d\omega}{dk} = v_g}$

si $\omega(k)$ no es lineal

Igualdad a demostrar

Consideremos la superposición de n ondas:

$$\Psi(x,t) = \sum_{n=1}^{n} A_n e^{i(k_n x - \omega_n t)}$$

Para $n \longrightarrow \infty$: (paso al continuo)

$\omega_n \rightarrow \omega$

$k_n \rightarrow k$ (variable continua)

$\sum_n \rightarrow \int dk$

$A_n \rightarrow A(k) \equiv$ distribución espectral

$$\Psi(x,t) = \int_0^{\infty} A(k)\, e^{i(kx - \omega t)} dk$$

$k^2 = \frac{\omega^2}{v^2}$, $\boxed{k = \pm \frac{\omega}{v}}$

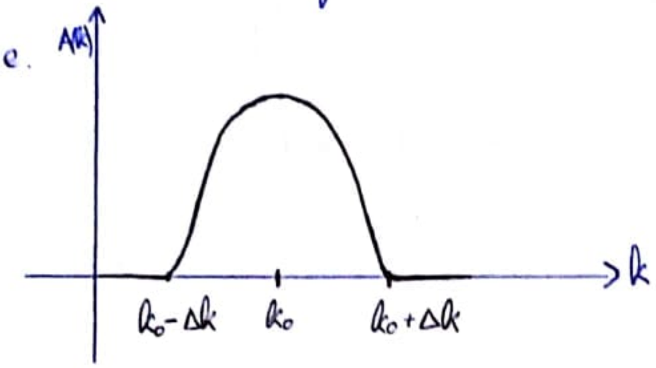
↳ k puede ser positivo o negativo

$$\boxed{\Psi(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)}\, dk}$$ Paquete de ondas

Velocidad de grupo del paquete de ondas:

Consideremos el caso en el que $A(k)$ toma valores distintos de cero entre $k_0 - \Delta k$ y $k_0 + \Delta k$.

p.e.



$$\left\{ \Psi(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)}\, dk = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k)\, e^{i(kx-\omega t)}\, dk \right\}$$

$\omega(k)$ se conoce como relación de dispersión

$\omega(k_0) = \omega_0$

Desarrollo de Taylor en torno a k_0:

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k_0} (k-k_0) + \dots \quad \approx \text{® suponemos } (k-k_0) \text{ pequeño}$$

$$\approx \omega_0 + \omega'(k_0)(k-k_0)$$

$$kx - \omega t \approx kx - \left[\omega_0 + \omega'(k_0)(k-k_0)\right] t = kx - \omega_0 t - \omega'(k_0)(k-k_0)t =$$

$$= kx - k_0 x + k_0 x - \omega_0 t - \omega'(k_0)(k-k_0)t =$$

$$= (k_0 x - \omega_0 t) - \omega'(k_0)(k-k_0)t + (k-k_0)x =$$

$$= (k_0 x - \omega_0 t) + (k-k_0)(x - \omega'(k_0)t)$$

↳ llevamos esto a la superposición del paquete de ondas:

$$\Psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} dk\, A(k)\, e^{i(kx-\omega t)} \underset{(k-k_0)\text{ pequeño}}{\simeq} \int_{k_0-\Delta k}^{k_0+\Delta k} dk \underbrace{A(k)\, e^{i(k-k_0)(x-\omega'(k_0)t)}}_{\substack{\text{onda que oscila}\\ \text{lentamente con } \lambda \\ \text{grande}}} \cdot \underbrace{e^{i(k_0 x-\omega_0 t)}}_{\substack{\text{oscilación rápida,}\\ \text{longitud de onda}\\ \text{pequeña}}}$$

↓

Amplitud Modulada del Paquete

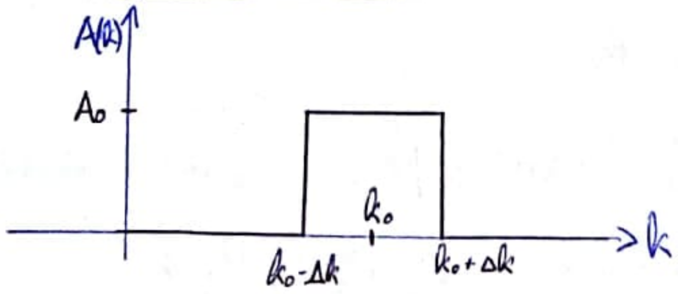
Velocidad de grupo ≡ Velocidad de fase de la amplitud modulada

Fase: $x - \omega'(k_0)t = \text{cte}$

$$dx - \omega'(k_0)\,dt = 0$$

$$\frac{dx}{dt} = \omega'(k_0) = \boxed{v_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}}$$

Ejemplo: Paquete cuadrado:



$$\varphi(x,t) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{i(kx-\omega t)} \simeq \int_{k_0-\Delta k}^{k_0+\Delta k} dk\, A_0\, e^{i(k-k_0)(x-\omega'(k_0)t)}\, e^{i(k_0x-\omega_0 t)} =$$

$$= A_0\, e^{i(k_0x-\omega_0 t)} \int_{k_0-\Delta k}^{k_0+\Delta k} dk\, e^{i(k-k_0)(x-\omega'(k_0)t)} =$$

$$= A_0\, e^{i(k_0x-\omega_0 t)} \frac{1}{i(x-\omega'(k_0)t)} \left[e^{i(k-k_0)(x-\omega'(k_0)t)} \right]_{k_0-\Delta k}^{k_0+\Delta k} =$$

$$= A_0\, e^{i(k_0x-\omega_0 t)} \frac{1}{i(x-\omega'(k_0)t)} \left[e^{i\Delta k(x-\omega'(k_0)t)} - e^{-i\Delta k(x-\omega'(k_0)t)} \right] =$$

$$= 2A_0 \frac{e^{i(k_0x-\omega_0 t)}}{[x-\omega'(k_0)t]} \left[\frac{e^{i\Delta k(x-\omega'(k_0)t)} - e^{-i\Delta k(x-\omega'(k_0)t)}}{2i} \right] =$$

$$= 2A_0\,\Delta k\, \frac{\sin[\Delta k(x-\omega'(k_0)t)]}{\Delta k(x-\omega'(k_0)t)}\, e^{i(k_0x-\omega_0 t)}$$

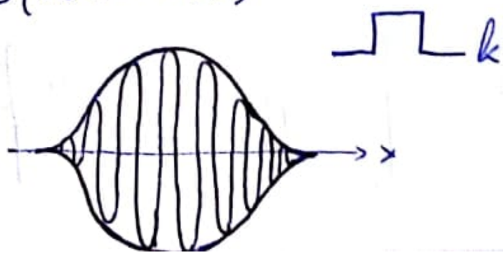
$$y = \Delta k(x-\omega'(k_0)t)$$

$$e^{i(k_0x-\omega_0 t)} = \cos(k_0x-\omega_0 t) + i\sin(k_0x-\omega_0 t)$$

Tomando la parte real:

$$\varphi(x,t) = \underbrace{2A_0\,\Delta k\,\frac{\sin y}{y}}_{\text{Amplitud modulada}} \cos(k_0x-\omega_0 t)$$

$$\boxed{\operatorname{sinc} y \equiv \frac{\sin y}{y}}$$



Si $\boxed{\Delta k \to 0} \Rightarrow \quad y \to 0 \quad \Rightarrow \underbrace{\lim_{y \to 0} \text{sinc}(y) = 1}$

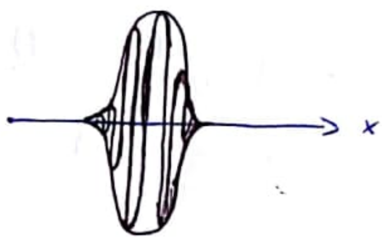
$$\varphi(x,t) = 2 A_0 \Delta k \cos(k_0 x - \omega_0 t)$$

$\rightarrow$ Distribución espectral muy estrecha en k (Δk pequeño)

$\Updownarrow$

Paquete de ondas muy deslocalizado en x.

Al contrario :



Si la distribución espectral es muy ancha en k (Δk grande) $\Leftrightarrow$ El paquete de ondas está muy localizado en x

$\rightarrow$ Ondas en 3 Dimensiones : $\boxed{\text{ondas planas}}$

(1D) $$\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \varphi}{\partial t^2}$$

(3D) $$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \varphi}{\partial t^2}$$

$$\boxed{\nabla^2 \varphi = \frac{1}{v^2} \frac{\partial^2 \varphi}{\partial t^2}} \longrightarrow$$ Separación de variables en 3D:

$$\varphi(x,y,z,t) = X(x)\, Y(y)\, Z(z)\, T(t)$$

$$\frac{\partial^2\varphi}{\partial x^2} = X''YZT \qquad \frac{\partial^2\varphi}{\partial y^2} = XY''ZT \qquad \frac{\partial^2\varphi}{\partial z^2} = XYZ''T$$

$$\frac{\partial^2\varphi}{\partial t^2} = XYZT''$$

$$\nabla^2\varphi = \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} \longrightarrow X''YZT + XY''ZT + XYZ''T = \frac{1}{v^2}XYZT''$$

$$\left(\cdot\frac{1}{XYZT}\right.$$

$$\boxed{\underbrace{\frac{X''}{X}}_{\text{solo depende de } x} + \underbrace{\frac{Y''}{Y}}_{\text{solo depende de } y} + \underbrace{\frac{Z''}{Z}}_{\text{solo depende de } z} = \underbrace{\frac{1}{v^2}\frac{T''}{T}}_{\text{solo depende de } t}}$$

① $$\boxed{\frac{T''}{T} = -\underbrace{\omega^2}_{\text{constante de separación}}}$$

$\circledast$ Def: $\vec{k} = (k_x, k_y, k_z)$

$|\vec{k}|^2 = k^2 = \dfrac{\omega^2}{v^2}$

$$\underbrace{\frac{X''}{X}} + \underbrace{\frac{Y''}{Y}} + \underbrace{\frac{Z''}{Z}} = -\frac{\omega^2}{v^2} \equiv -k^2 = \overbrace{\underbrace{-k_x^2}\underbrace{-k_y^2}\underbrace{-k_z^2}}^{\text{constantes de separación}}$$

$X''/X = -k_x^2$ ②

$Y''/Y = -k_y^2$ ③

$Z''/Z = -k_z^2$ ④

Las soluciones son 4 osciladores armónicos:

① $X(x) = Ae^{ik_x x} + Be^{-ik_x x}$ ② $Y(y) = Ce^{ik_y y} + De^{-ik_y y}$

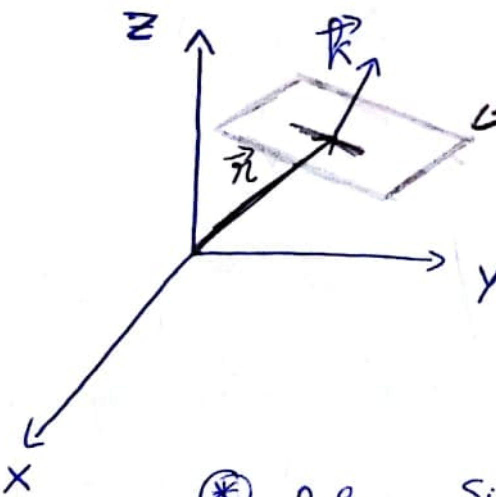
③ $Z(z) = E e^{ik_z z} + F e^{-ik_z z}$ ④ $T(t) = G e^{i\omega t} + H e^{-i\omega t}$

$$\Psi(x,y,z,t) \sim e^{i(\pm k_x x \pm k_y y \pm k_z z - \omega t)} \sim e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

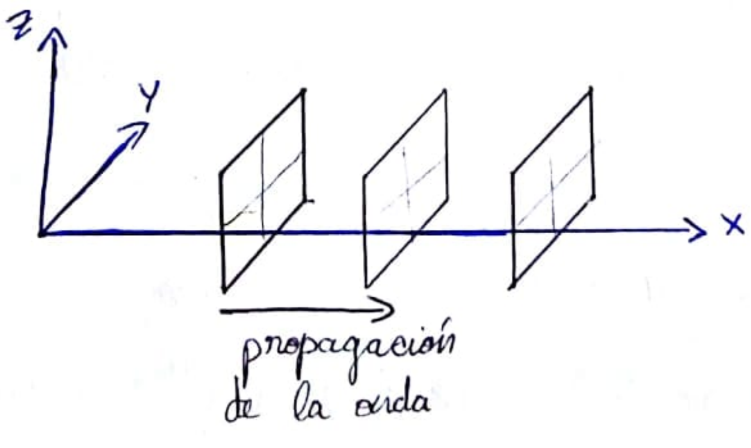
Se trata de una <u>onda plana</u>:

Para $t = t_0 = cte$ el lugar geométrico de los puntos en los que $\Psi(x,y,z,t_0) = cte$ verifica que $\vec{k}\cdot\vec{r} = cte \Rightarrow \underbrace{k_x x + k_y y + k_z z = cte}$

ecuación de un plano perpendicular al vector $\vec{k}$ → dirección de propagación de la onda



㊍ p.e. si $\vec{k} = (k_x, 0, 0)$:



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4. Ondas

4.1. Por un medio unidimensional propagase unha onda sonora de 440 Hz no sentido negativo do eixo x cunha velocidade de 330 ms^{-1} e amplitude A. Escribir a solución harmónica correspondente. ¿Queda totalmente definida a onda cos datos aportados? Se ademais se propaga no mesmo sentido outra onda análoga de igual amplitude e frecuencia 441 Hz, calcular a onda resultante e representar esquemáticamente as oscilacións sonoras nun punto fixo en función do tempo. ¿Qué frecuencia ten o sonido escoitado por un observador nese punto? ¿É a intensidade uniforme? Comentar cualitativamente como se modifica o resultado cando a frecuencia da segunda onda se acerca ou aparta do valor 440 Hz, e cando as amplitudes sexan diferentes. Buscar a relación coa afinaciòn de instrumentos musicais.

4.2. Comprobar se as seguintes funcións representan unha onda viaxeira con pefil constante: $\Psi = A\exp\left[-(x-4t)^2\right]$, $\Psi = A\exp\left[-x\right]\exp\left[-(x-4t)^2\right]$, $\Psi = A\exp\left[-(x^2-4t^2)\right]$, $\Psi = A\cos\left(x^2-t^2\right)$.

4.3. Unha corda vibrante de lonxitude l remata nun dos extremos ($x = 0$) nun anel de masa desprezable que esvara sen rozamento por unha variña vertical. Coa ecuación do movemento do anel xustificar que a condición de contorno da corda no extremo ven dada por:

$$\left[\frac{\partial u}{\partial x}\right]_{x=0} = 0.$$

Se o extremo $x = l$ está fixo, atopar os modos normais de vibración.

4.4. **a)** Establecer a condición de contorno para a corda do problema anterior supoñendo que o anel ten masa m. **b)** Estudiar a reflexión dunha onda que incida en $x = 0$. Calcular o coeficiente de reflexión e a proporción de enerxía reflectida. **c)** Repetir o apartado b) para un anel de masa desprezable pero sometido a unha forza de rozamento dada por $F_r = -\gamma v$, sendo v é a velocidade do anel. ¿Pódese conseguir un extremo "perfecto" da corda na que non haxa onda reflectida? **d)** Considerar oscilacións excitadas dende o outro extremo. ¿Formaríanse ondas estacionarias? ¿Necesitaríamos aportar enerxía? Comprobar que o fluxo neto no extremo da corda coincide coa enerxía disipada no anel.

4.5. Bátese unha corda de piano de lonxitude l, tensión τ e densidade μ, suxeita por ámbolos dous extremos, a unha distancia a de un deles, cun mazo de piano de masa m e velocidade v_0. Supoñer que o mazo colisiona elásticamente cun pedazo pequeno Δl de corda centrada en $x = a$. Obter o movemento da corda utilizando o método de separación variables. Se se desexa que non esté presente o sétimo harmónico da frecuencia fundamental, ¿en que punto $x = a$ debe o mazo bater a corda?

4.6. Un pulso cadrado de amplitude unidade e anchura a está representado no instante $t = 0$ pola función:

$$\Psi = \begin{cases} 1, & \text{se } |x| < \frac{a}{2}; \\ 0, & \text{se } \frac{a}{2} < |x| < L. \end{cases}$$

que se repite por todo o espacio con periodicidade $2L$. **(a)** Calcular as compoñentes de Fourier do pulso. **(b)** Representar gráficamente o pulso en función de x en $t = 0$ e as amplitudes das compoñentes de Fourier en función de k. **(c)** Se mantemos o mesmo pulso pero facemos que sexa periódico en $4L$ ¿como se modifica o resultado? ¿Que acontece cos valores de k das compoñentes de Fourier? **(d)** Se polo contrario facemos o pulso máis estreito $a/2$ ¿cómo se modifica agora o resultado?

4.7. Calcular os modos normais de oscilación dunha membrana elástica sometida a tensión uniforme por un marco ríxido rectangular de lados a e b. Debuxar as líñas nodais para algúns casos. ¿Que acontece se $a = b$? Representar as líñas nodais e a oscilación para a combinación de dúas oscilacións de igual amplitude correspondentes ós modos normais 2 e 3 cando se ordean de frecuencia menor a maior.

4.8. As ondas do mar sofren dispersión cando a profundidade é grande en comparación coa lonxitude de onda, polo que a súa forma se altera durante a propagación. A correspondente relación de dispersión ven dada por $\omega^2 = gk$. Un barco a motor avanza cara a praia. Se o grupo de ondas orixinado xunto á proa do barco se propaga ata a praia cunha velocidade de 10 m/s, ¿cal é a velocidade dunha onda nese grupo que teña exactamente $\lambda = 1$ m?.

4.9. Cando ademais da gravidade se considera a tensión superficial da auga ($\tau = 0.074\ N/m$), a relación de dispersión das ondas en augas profundas convértese en $\omega^2 = gk + \tau k^3/\rho$ ($\rho = 1\ g/cm^3$). **(a)** Representar gráficamente ω frente a k. **(b)** ¿Para que lonxitudes de onda a tensión superficial é importante? **(c)** ¿É un medio dispersivo? **(d)** Calcular as lonxitudes de onda para as que as velocidades de fase e de grupo son mínimas.

Boletín : Ondas

① Medio unidimensional

Onda sonora $\longrightarrow f_1 = 440\,Hz$ A

Sentido negativo del eje $x \longrightarrow v = 330\, m/s$

¿Solución armónica? ¿Queda totalmente definida la onda?

$$k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi f_1}{v} \longrightarrow k_1 = \frac{2\pi\,(440\,s^{-1})}{330\,m\,s^{-1}} = \frac{8\pi}{3}\,m^{-1} \simeq 8\,m^{-1}$$

$$\omega_1 = \frac{2\pi}{T_1} = 2\pi f_1 \longrightarrow \omega_1 = 2\pi\,(440\,s^{-1}) = 880\pi\,s^{-1} \simeq 2765\,s^{-1}$$

$$u_1(x,t) = A\, e^{-i(k_1 x + \omega_1 t)}$$

$$y_1(x,t) = A\cos(k_1 x + \omega_1 t)$$

No está definida totalmente (no se nos indica su fase inicial).

Además, se propaga en el mismo sentido una onda de igual amplitud A y $f_2 = 441\,Hz$.

¿Onda resultante?

$$u_2(x,t) = A\, e^{-i(k_2 x + \omega_2 t)}$$

$$y_2(x,t) = Re\left[A\, e^{-i(k_2 x + \omega_2 t)}\right] = A\cos(k_2 x + \omega_2 t)$$

$$y(x,t) = y_1(x,t) + y_2(x,t) = A\left[\cos(k_2 x + \omega_2 t) + \cos(k_1 x + \omega_1 t)\right] =$$

$$= 2A\cos\left[\frac{(k_1+k_2)x + (\omega_1+\omega_2)t}{2}\right]\cos\left[\frac{(k_2-k_1)x + (\omega_2-\omega_1)t}{2}\right]$$

$$\begin{array}{l} \cos(A+B) = \cos A\cos B - \sin A\sin B \\ + \ \cos(A-B) = \cos A\cos B + \sin A\sin B \\ \hline \cos(A+B) + \cos(A-B) = 2\cos A\cos B \end{array}$$

$$A = \frac{(k_2+k_1)x + (\omega_2+\omega_1)t}{2}, \quad B = \frac{(k_2-k_1)x + (\omega_2-\omega_1)t}{2}$$

Podemos tener en cuenta que si $f_2 > f_1$: $f_2 = f_1 + \Delta f$

$$\omega_2 = 2\pi f_2 = 2\pi f_1 + 2\pi \Delta f = \omega_1 + \Delta\omega$$

$$k_2 = \frac{2\pi f_2}{v} = \frac{2\pi f_1}{v} + \frac{2\pi \Delta f}{v} = k_1 + \Delta k$$

En nuestro problema, $\Delta f = 1\,s^{-1}$, $f_1 = 440\,s^{-1}$, $\Delta f \ll f_1$

Podemos escribir la solución anterior como:

$$y(x,t) = 2A \cos\left(\frac{(2k_1+\Delta k)}{2}x + \frac{(2\omega_1+\Delta\omega)}{2}t\right)\cos\left(\frac{\Delta k}{2}x + \frac{\Delta\omega}{2}t\right)$$

$$\boxed{y(x,t) = 2A\cos\left(\frac{\Delta k x + \Delta\omega t}{2}\right)\cos\left[\left(k_1+\frac{\Delta k}{2}\right)x + \left(\omega_1 + \frac{\Delta\omega}{2}\right)t\right]}$$

Representar esquemáticamente las ondas sonoras en un punto fijo:

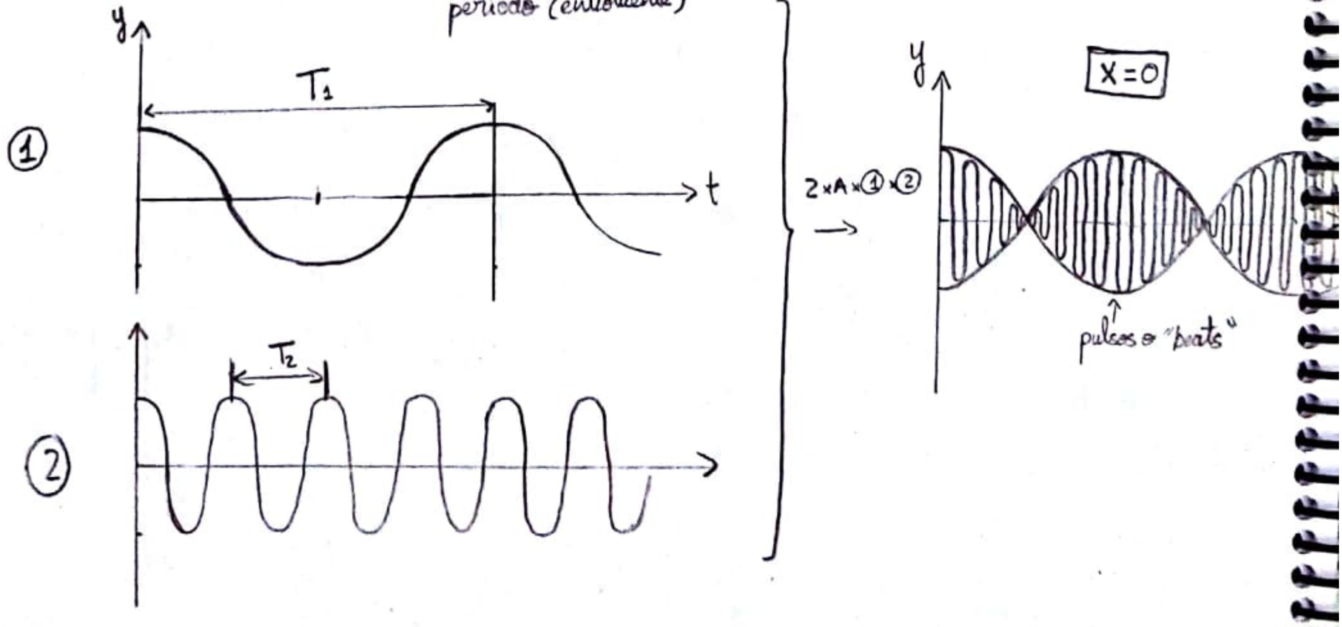
Elegimos por simplicidad el punto $x=0$:

$$y(x=0,t) = 2A\cos\left(\frac{\Delta\omega}{2}t\right)\cos\left(\left(\omega_1+\frac{\Delta\omega}{2}\right)t\right)$$

$$\boxed{\begin{aligned}\omega_1 &= 880\pi\,s^{-1}\\ \Delta\omega &= 2\pi\,s^{-1}\end{aligned}}$$

$$y(x=0,t) = 2A\,\underbrace{\cos(\pi t\,s^{-1})}_{①}\cdot\underbrace{\cos(881\pi t\,s^{-1})}_{②}$$

Mucho mayor período (envolvente)



4Ejs

¿Qué frecuencia tiene el sonido escuchado por un observador en dicho punto?

La frecuencia del sonido escuchado es con la que llegan los "beats" al observador, es decir la frecuencia de la envolvente:

$f_{env} = ?$ $\qquad \boxed{f_{env} = \frac{\omega_{env}}{2\pi} = \frac{\Delta\omega}{2\pi} = \Delta f = \boxed{1\,Hz}}$

¿Es uniforme la intensidad? No, varía con la amplitud de la envolvente.

¿Cómo se modifica el resultado si las frecuencias son muy similares o muy distintas?

→ Frecuencias iguales → $f_2 \to f_1$ ⇒ $\boxed{\varphi(0,t) \to 2A\cos\omega_1 t}$

$(\Delta\omega \to 0)$

→ Frecuencias muy distintas → $f_2 \gg f_1$ ⇒ $\omega_2 \gg \omega_1$

$$\cos\left(\frac{\Delta\omega}{2}t\right) = \cos\left(\frac{\omega_2-\omega_1}{2}t\right) \approx \cos\left(\frac{\omega_2}{2}t\right)$$

$$\cos\left[\left(\omega_1 + \frac{\Delta\omega}{2}\right)t\right] = \cos\left[\left(\omega_1 + \frac{\omega_2}{2} - \frac{\omega_1}{2}\right)t\right] \approx \cos\left(\frac{\omega_2}{2}t\right)$$

$$\boxed{\varphi(0,t) \approx 2A\cos^2\left(\frac{\omega_2}{2}t\right)}$$

¿Cómo es el resultado si las amplitudes son distintas?

Relación con la afinación de los instrumentos musicales:

El oído humano no es capaz de distinguir unos pocos Hz en torno a 440 Hz con objeto de afinar un instrumento musical.

Sí puede oír la envolvente, las variaciones en la amplitud de frecuencia pequeña que tienen lugar al superponer la onda del instrumento no afinado y la del diapasón afinado. Cuanto menor sea la frecuencia de la envolvente $(\nu_2 - \nu_1)$, mejor afinado estará nuestro instrumento.

② Comprobar si las siguientes funciones representan una onda viajera con perfil constante:

a) $\psi = A\, e^{[-(x-4t)^2]} \longrightarrow f(x-vt)$ con $v = 4 \Rightarrow$ Sí

b) $\psi = A\, e^{-x} e^{-(x-4t)^2} \rightarrow$ No es de la forma $f(x-vt) + g(x+vt)$
$\Downarrow$ No

c) $\psi = A\, e^{-(x^2-4t^2)} = A\, e^{-(x+2t)(x-2t)} \rightarrow$ No es combinación lineal de una función de $x-vt$ y una de $x+vt$
$\Downarrow$ No

Podemos sustituir en la ec. de ondas y ver que no la verifica.

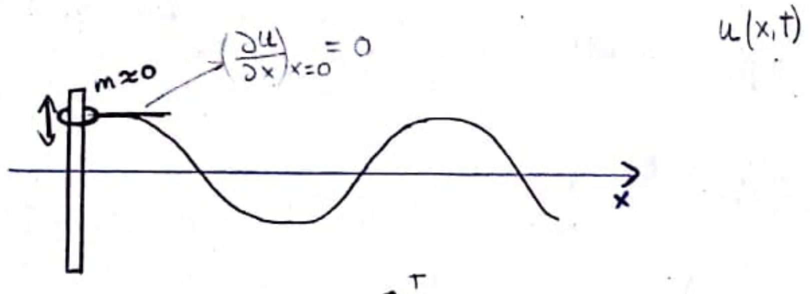
d) $\psi = A\cos(x^2-t^2) \rightarrow$ No de la forma $f(x+vt) + g(x-vt)$ $\Rightarrow$ No

③ Cuerda vibrante longitud ℓ

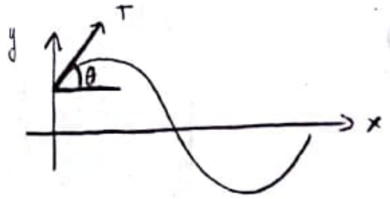
anillo de masa despreciable en un extremo $(x = 0)$

→ resbala sin rozamiento por una barra vertical

→ Usar la ecuación de movimiento del anillo para justificar la condición de contorno de la cuerda en su extremo: $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$



Demostración:



2ª Ley de Newton en $x = 0$:

$$F_y = T \cdot \sin\theta = m \cdot a_y \overset{(m \approx 0)}{\approx} 0$$

Definición de derivada

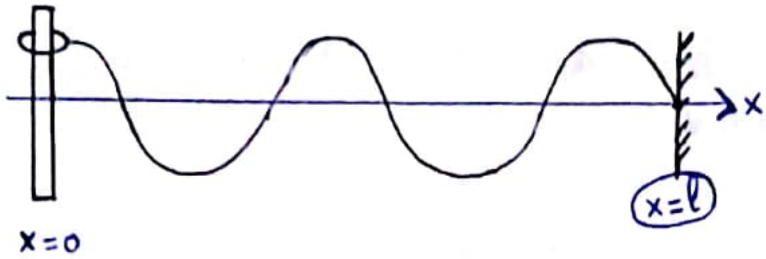
$$\theta \ll 1 \Rightarrow \sin\theta \simeq \tan\theta = \left.\frac{\partial u}{\partial x}\right|_{x=0}$$

$$F_y = T \cdot \sin\theta \simeq T \cdot \tan\theta = \left.\frac{\partial u}{\partial x}\right|_{x=0} \simeq 0$$

$$\Downarrow$$

$$\boxed{\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0}$$

Si el extremo $x = \ell$ está fijo. encontrar los modos normales de vibración:



$u_x(0,t) = 0$ $\qquad\qquad$ $u(\ell,t) = 0$

$$u(x,t) = A\, e^{i(kx-\omega t)} + B\, e^{i(-kx-\omega t)}$$

(Solución general de la ec. ondas)
↳ combinación lineal de dos ondas. una hacia la izquierda y otra hacia la derecha

$$\frac{\partial u}{\partial x} = ikA e^{i(kx-\omega t)} - ikB e^{i(-kx-\omega t)}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = ikA e^{-i\omega t} - ikB e^{-i\omega t} = 0 \quad \forall t$$

$$ike^{-i\omega t}(A - B) = 0 \quad \forall t$$

$$A - B = 0 \,, \quad \boxed{A = B}$$

$$u(x,t) = A e^{i(kx-\omega t)} + A e^{i(-kx-\omega t)} = A e^{-i\omega t}\left[e^{ikx} + e^{-ikx}\right] =$$

$$= A e^{-i\omega t}(2\cos(kx)) = 2A\, e^{-i\omega t}\cos kx$$

$$u(\ell,t) = 2A\, e^{-i\omega t}\cos k\ell = 0 \quad \forall t$$

$$\hookrightarrow \cos k\ell = 0 \Rightarrow k\ell = (2n+1)\frac{\pi}{2}$$

$$\boxed{k_n = \frac{(2n+1)\pi}{2\ell}}$$

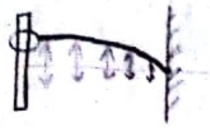
$$\lambda_n = \frac{2\pi}{k_n}$$

Longitudes de onda permitidas

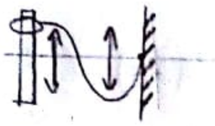
$$\boxed{\lambda_n = \frac{4\ell}{2n+1}} \quad n = 0,1,2,3\ldots \qquad \boxed{\omega_n = v k_n}$$

Modos normales:

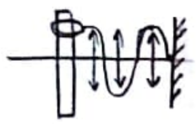
$n = 0 \longrightarrow \lambda_0 = 4l \,, \quad l = \frac{1}{4}\lambda_0$



$n = 1 \longrightarrow \lambda_1 = \frac{4}{3} l \,, \quad l = \frac{3}{4}\lambda_1$



$n = 2 \longrightarrow \lambda_2 = \frac{4}{5} l \,, \quad l = \frac{5}{4}\lambda_2$



④ Mismo problema que el anterior pero ahora el anillo tiene masa m.

a) Condición de contorno ?

2ª Ley de Newton: $F_y = T\sin\theta = m a_y$

$$\left.\begin{aligned} F_y &= T\sin\theta \underset{\theta \ll 1}{\approx} T\tan\theta = T \left.\frac{\partial u}{\partial x}\right|_{x=0} \\ m a_y &= m \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=0} \end{aligned}\right\}$$

C. contorno

$$\boxed{T \left.\frac{\partial u}{\partial x}\right|_{x=0} = m \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=0}}$$

$$\boxed{u(x,t) = \underbrace{A e^{i(\omega t + kx)}}_{\text{onda que viaja hacia la izquierda}} + \underbrace{B e^{i(\omega t - kx)}}_{\text{onda que viaja hacia la derecha}}}$$

(Solución general a la ecuación de ondas)

b) Estudiar la reflexión de una onda que incida en $x=0$, $\hookrightarrow \xleftarrow{\text{inc.}} \quad \xrightarrow{\text{refl}}$

$$u(x,t) = \overset{\xleftarrow{\text{incidente}}}{A e^{i(kx+\omega t)}} + \overset{\xrightarrow{\text{reflejada}}}{B e^{i(-kx+\omega t)}}$$

$$\frac{\partial u}{\partial x} = ikA e^{i(kx+\omega t)} - ikB e^{i(-kx+\omega t)}, \quad \boxed{\left.\frac{\partial u}{\partial x}\right|_{x=0} = ik e^{i\omega t}(A-B)}$$

$$\frac{\partial u}{\partial t} = i\omega A e^{i(kx+\omega t)} + i\omega B e^{i(-kx+\omega t)}$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 A e^{i(kx+\omega t)} - \omega^2 B e^{i(-kx+\omega t)}, \quad \boxed{\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=0} = -\omega^2 e^{i\omega t}(A+B)}$$

$$\boxed{T \left.\frac{\partial u}{\partial x}\right|_{x=0} = m \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=0}} \qquad ikTe^{i\omega t}(A-B) = -m\omega^2 e^{i\omega t}(A+B) \quad \forall t$$

$$\boxed{ikT(A-B) = -m\omega^2(A+B)}$$

amplitud de la onda reflejada ($\longrightarrow$)

$$\boxed{r = \frac{B}{A} \equiv \text{coeficiente de reflexión}}$$

$$ikT\left(1 - \frac{B}{A}\right) = -m\omega^2\left(1 + \frac{B}{A}\right)$$

$$ikT + m\omega^2 = \frac{B}{A}\left(ikT - m\omega^2\right)$$

$$r = \frac{B}{A} = \frac{m\omega^2 + ikT}{ikT - m\omega^2} = -\frac{m\omega^2 + ikT}{m\omega^2 - ikT} \overset{\circledast}{=}$$

$\circledast$ Definimos $z \equiv m\omega^2 + ikT$
y $z^* \equiv m\omega^2 - ikT$ (complejos conjugados)

$$= -\frac{z}{z^*} = -\frac{|z|\,e^{i\varphi}}{|z|\,e^{-i\varphi}} = -e^{2i\varphi}, \qquad \boxed{r = -e^{2i\varphi}, \text{ con } \varphi = \arctan\frac{kT}{m\omega^2}}$$

$R = \left|\frac{B}{A}\right|^2 \equiv$ proporción de energía reflejada $\equiv \frac{\text{energía reflejada}}{\text{energía incidente}}$

$$\boxed{R = \left|\frac{B}{A}\right|^2 = \left|-e^{2i\varphi}\right|^2 = 1} \longrightarrow \boxed{\text{Reflexión total}}$$

c) $m \simeq 0$

Rozamiento en el anillo, $F_R = -\gamma v$ → velocidad del anillo

Estudiar la reflexión (como en B)

¿Se puede conseguir un extremo "perfecto" de la cuerda de forma que no haya reflexión?

→ C. contorno:

2ª Ley de Newton: $\boxed{F_y = m a_y}$

$$F_y = T\sin\theta - \gamma v \approx T\tan\theta - \gamma v = $$
$$= T\left.\frac{\partial u}{\partial x}\right|_{x=0} - \gamma\left.\frac{\partial u}{\partial t}\right|_{x=0}$$

$m a_y \simeq 0$

$$\boxed{T\left.\frac{\partial u}{\partial x}\right|_{x=0} = \gamma\left.\frac{\partial u}{\partial t}\right|_{x=0}}$$

$$u(x,t) = \overbrace{A\,e^{i(\omega t+kx)}}^{\longleftarrow} + \overbrace{B\,e^{i(\omega t-kx)}}^{\longrightarrow}$$

$$\frac{\partial u}{\partial x} = Aik\,e^{i(\omega t+kx)} - Bik\,e^{i(\omega t-kx)}, \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = ik\,e^{i\omega t}(A-B)$$

$$\frac{\partial u}{\partial t} = Ai\omega\,e^{i(\omega t+kx)} + Bi\omega\,e^{i(\omega t-kx)}, \quad \left.\frac{\partial u}{\partial t}\right|_{x=0} = i\omega\,e^{i\omega t}(A+B)$$

$$\boxed{T\left.\frac{\partial u}{\partial x}\right|_{x=0} = \gamma\left.\frac{\partial u}{\partial t}\right|_{x=0}} \qquad ikT(A-B)\,e^{i\omega t} = i\omega\gamma(A+B)\,e^{i\omega t} \quad \forall t$$

$$\boxed{kT(A-B) = \omega\gamma(A+B)}$$

$$r = \frac{B}{A}$$ → onda reflejada ($\longrightarrow$)

$$kT(A-B) = \gamma\omega(A+B)$$

$$kT\left(1-\frac{B}{A}\right) = \gamma\omega\left(1+\frac{B}{A}\right)$$

$$-\frac{B}{A}(kT+\gamma\omega) = \gamma\omega - kT$$

$$r = \frac{B}{A} = \frac{kT-\gamma\omega}{kT+\gamma\omega}$$

reflectividad

$$\boxed{R = \left|\frac{B}{A}\right|^2 = \left(\frac{kT-\gamma\omega}{kT+\gamma\omega}\right)^2}$$

Casos:

- Si $\gamma = 0 \implies R = 1 \implies$ Reflexión total
- Si $\gamma \neq 0 \implies R < 1 \implies$ Energía perdida en el rozamiento

⊛ Caso particular:

Si $kT = \gamma\omega$, $\gamma = \frac{kT}{\omega} = \frac{T}{v}$ ($v = \frac{\omega}{k}$)

$\Downarrow$ $\boxed{R = 0}$ → No hay reflexión: Toda la energía incidente se convierte en rozamiento (calor) → o es absorbida

Este es el caso que buscábamos de extremo "perfecto".

Demostremos que en este caso la potencia incidente y disipada coinciden: $P_{inc} \overset{?}{=} P_{dis.}$

$$P_{inc}\Big|_{x=0} = \vec{F}\cdot\frac{d\vec{r}}{dt}\Big|_{x=0} = \vec{F}\cdot\vec{v}\Big|_{x=0} = T\left(\frac{\partial u}{\partial x}\Big|_{x=0}\right)\left(\frac{\partial u}{\partial t}\Big|_{x=0}\right) =$$

→ velocidad del anillo (coincide en módulo con la de la onda).

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$$= T\left(Re\left[ike^{i\omega t}(A-B)\right]\right)\left(Re\left[i\omega e^{i\omega t}(A+B)\right]\right) =$$

$$= -Tk\omega(A^2-B^2)\left[(-\sin\omega t)(-\sin\omega t)\right] = -Tk\omega(A^2-B^2)\sin^2\omega t$$

$$\langle P_{inc}\rangle = Tk\omega(A^2-B^2)\langle\sin^2\omega t\rangle \longleftarrow$$ Promedio en un período

$$\langle\sin^2\omega t\rangle = \frac{\int_0^T \sin^2\omega t\,dt}{T} = \frac{\int_0^T \frac{1-\cos 2\omega t}{2}\,dt}{T} = \frac{[t]_0^T - \left[\frac{\sin(2\frac{2\pi}{T}t)}{2}\right]_0^T}{2T} =$$

$$= \frac{T}{2T} = \frac{1}{2}$$

$$|\langle P_{inc}\rangle| = \frac{1}{2}kT\omega(A^2-B^2) = \frac{1}{2}kT\omega A^2(1-r^2)$$

® Si $r=1 \Rightarrow$ reflexión total $\Rightarrow$ $\langle P_{inc}\rangle = 0$ (en el anillo no se realiza trabajo porque toda la energía se refleja)

↳ ondas estacionarias

Si hay rozamiento, se pierde energía en el anillo:

$$P_{dis} = \vec{F}_r\cdot\vec{v} = -\gamma v^2 = -\gamma\left(\left.\frac{\partial u}{\partial t}\right|_{x=0}\right)^2 =$$

$$= -\gamma\left[Re\left(i\omega e^{i\omega t}(A+B)\right)\right]^2 = -\gamma\omega^2(A+B)^2[-\sin\omega t]^2 =$$

$$= -\gamma\omega^2(A+B)^2\sin^2\omega t$$

Promediando en un período:

$$\langle P_{dis}\rangle = -\gamma\omega^2(A+B)^2\langle\sin^2\omega t\rangle$$

$$|\langle P_{dis}\rangle| = \frac{1}{2}\gamma\omega^2(A+B)^2 = \frac{1}{2}\gamma\omega^2A^2(1+r)^2$$

Comparamos los valores de las potencias incidente y disipada en el anillo:

$$\boxed{\frac{\langle P_{inc} \rangle}{\langle P_{dis} \rangle}} = \frac{\frac{1}{2} kT\omega A^2(1-r^2)}{\frac{1}{2}\gamma\omega^2 A^2(1+r)^2} = \frac{kT}{\gamma\omega}\frac{(1-r)(1+r)}{(1+r)(1+r)} =$$

$$= \frac{kT}{\gamma\omega}\frac{1-r}{1+r} = \frac{kT}{\gamma\omega}\frac{1-\frac{kT-\gamma\omega}{kT+\gamma\omega}}{1+\frac{kT-\gamma\omega}{kT+\gamma\omega}} =$$

$$= \frac{kT}{\gamma\omega}\frac{(kT+\gamma\omega)-(kT-\gamma\omega)}{(kT+\gamma\omega)+(kT-\gamma\omega)} =$$

$$= \frac{kT}{\gamma\omega}\frac{2\gamma\omega}{2kT} = \boxed{1}$$

$$\frac{kT}{\gamma\omega}\frac{1-r}{1+r} = 1$$

Donde vemos que si

$$\boxed{r=0}$$

$$\frac{kT}{\gamma\omega} = 1, \quad \boxed{kT = \gamma\omega}$$

q.e.d.

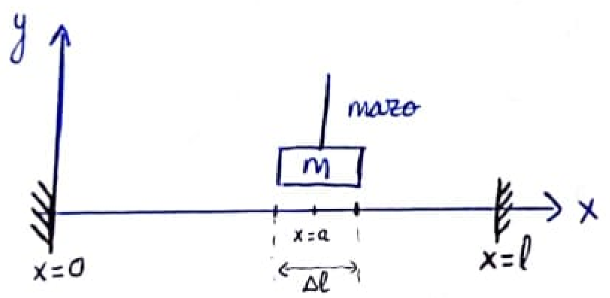
(4.5) Cuerda de piano
↳ sujeta por ambos extremos

longitud $\equiv l$
tensión $\equiv \tau$
densidad lineal de masa $\equiv \mu$

Golpeada a una distancia a de uno de los extremos con una $\underbrace{\text{masa } m}_{\text{mazo del piano}}$ a velocidad v_0

Choque elástico con un elemento de cuerda de longitud Δl centrado en $x = a$.

¿En qué punto $x = a$ debe golpear el mazo para que no aparezca el séptimo armónico de la frecuencia fundamental?



Choque elástico
↳ Conservación del momento lineal y la energía cinética:

· Momento lineal:

$$\boxed{m v_0 = m \underbrace{v_0'}_{\text{velocidad mazo}} + (\mu \Delta l) \underbrace{V}_{\text{velocidad cuerda}}} \quad (1)$$

(tras el choque)

· Energía cinética:

$$\boxed{\frac{1}{2} m v_0^2 = \frac{1}{2} m v_0'^2 + \frac{1}{2} (\mu \Delta l) V^2}$$

$$\boxed{m v_0^2 = m v_0'^2 + (\mu \Delta l) V^2} \quad (2)$$

Def: $\boxed{\alpha \equiv \frac{\mu \Delta l}{m}}$

$$\left.\begin{aligned} (1) \quad & v_0 = v_0' + \alpha V \\ (2) \quad & v_0^2 = v_0'^2 + \alpha V^2 \end{aligned}\right\}$$

$v_0' = v_0 - \alpha V \longrightarrow v_0^2 = (v_0 - \alpha V)^2 + \alpha V^2$

$$V_0^2 = V_0^2 + \alpha^2 V^2 - 2\alpha V_0 V + \alpha V^2$$

$$0 = V^2(\alpha^2+\alpha) - 2\alpha V_0 V \quad , \qquad 0 = V\alpha(1+\alpha) - 2\alpha V_0$$

$$\boxed{V = \frac{2V_0}{\alpha+1}} \longrightarrow$$ velocidad inicial tras el choque para el elemento de cuerda entre $a - \frac{\Delta l}{2}$ y $a + \frac{\Delta l}{2}$

Condiciones iniciales y de contorno para la cuerda:

→ Fija en los extremos : $y(0,t) = y(l,t) = 0 \quad \forall t$

→ Inicialmente en equilibrio : $y(x,0) = 0 \quad \forall x$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0}^{x} = \begin{cases} V = \frac{2V_0}{\alpha+1} , & x \in \left(a - \frac{\Delta l}{2}, a + \frac{\Delta l}{2}\right) \\ 0 & \text{en el resto de la cuerda} \end{cases}$$

Solución:

$$y(x,t) = A e^{i(kx-\omega t)} + B e^{-i(kx+\omega t)} = \left[A e^{ikx} + B e^{-ikx}\right] e^{-i\omega t}$$

① $y(0,t) = (A+B)e^{-i\omega t} = 0 \quad \forall t \Rightarrow A + B = 0 , \ \boxed{A = -B}$

$$y(x,t) = A\left[e^{ikx} - e^{-ikx}\right]e^{-i\omega t} = 2iA \sin kx \, e^{-i\omega t}$$

② $y(l,t) = 2Ai \sin kL \, e^{-i\omega t} = 0 \quad \forall t \Rightarrow \sin kL = 0 \Rightarrow kL = n\pi$

$$\boxed{k_n = \frac{n\pi}{L}}$$

$y_n(x,t) = 2A_n i \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-i\omega_n t}$ (forma de las soluciones)

$A_n = a_n + i b_n$ (complejo en general)

$$y_n(x,t) = 2i(a_n + ib_n) \sin\left(\frac{n\pi x}{L}\right)\left[\cos(\omega_n t) - i \sin(\omega_n t)\right]$$

$$\mathrm{Re}\left[y_n(x,t)\right] = 2a_n \sin\frac{n\pi x}{L} \sin\omega_n t - 2b_n \sin\frac{n\pi x}{L} \cos\omega_n t$$

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③ $y(x,0) = 0 \quad \forall x$

$y_n(x,0) = -2 b_n \sin \frac{n\pi x}{L} = 0 \quad \forall x \Rightarrow \boxed{b_n = 0}$

$y_n(x,t) = \underbrace{2 a_n}_{\substack{|||\\ B_n}} \sin \frac{n\pi x}{L} \sin \omega_n t = B_n \sin \frac{n \pi x}{L} \sin \omega_n t$

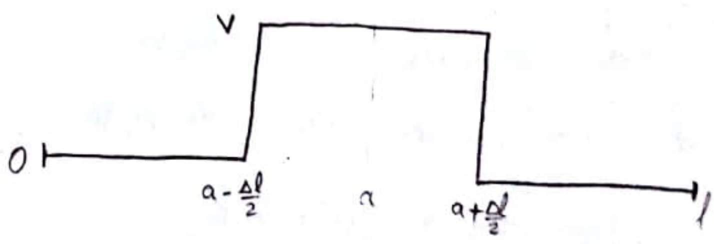
Solución general: $\boxed{y(x,t) = \sum_n B_n \sin\left(\frac{n\pi x}{L}\right) \sin \omega_n t}$

④ $\frac{\partial y}{\partial t}(x,0) = V = \frac{2 v_0}{\alpha + 1} \quad$ si $\quad x \in \left(a - \frac{\Delta l}{2}, a + \frac{\Delta l}{2}\right)$

$\frac{\partial y}{\partial t}(x,0) = 0 \quad$ en el resto

$$\frac{\partial y}{\partial t}(x,t) = \sum_n B_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t \cdot \omega_n$$

$$\frac{\partial y}{\partial t}(x,0) = \underbrace{\sum_n B_n \omega_n \sin \frac{n\pi x}{L}}_{\text{expansión en serie de Fourier de la función cuadrada}} = \underbrace{\begin{cases} V = \frac{2 v_0}{\alpha+1} & x \in \left(a - \frac{\Delta l}{2}, a + \frac{\Delta l}{2}\right) \\ 0 & \text{resto} \end{cases}}_{\text{función cuadrada}}$$



$$\omega_n B_n = \frac{1}{L} \int_{a-\frac{\Delta l}{2}}^{a+\frac{\Delta l}{2}} \frac{\partial y}{\partial t}(x) \sin \frac{n\pi x}{L} \, dx =$$

$$= \frac{1}{L} \int_{a-\frac{\Delta l}{2}}^{a+\frac{\Delta l}{2}} V \sin \frac{n\pi x}{L} \, dx = -\frac{V}{L} \frac{\left[\cos \frac{n\pi x}{L}\right]_{a-\frac{\Delta l}{2}}^{a+\frac{\Delta l}{2}}}{n\pi / L} = -\frac{V}{n\pi} \left[\cos \frac{n\pi}{L}\left(a + \frac{\Delta l}{2}\right) - \right.$$

$$\left. - \cos \frac{n\pi}{L}\left(a - \frac{\Delta l}{2}\right)\right]$$

(*) $\cos(\alpha+\beta) - \cos(\alpha - \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta -$

$- \cos\alpha \cos\beta - \sin\alpha \sin\beta = -2 \sin\alpha \sin\beta$

$$\omega_n B_n = -\frac{v}{n\pi}(-2)\ \sin\left(\frac{n\pi}{L}a\right)\ \sin\left(\frac{n\pi}{L}\frac{\Delta \ell}{2}\right) = \qquad \circledast\ k_n = \frac{\pi n}{L}$$

$$= \frac{2v}{Lk_n}\sin(k_n a)\sin\left(k_n\frac{\Delta\ell}{2}\right)$$

$$B_n = \frac{2v}{L k_n \omega_n}\sin(k_n a)\sin\left(k_n\frac{\Delta\ell}{2}\right)$$

$$\downarrow$$

$$y(x,t) = \sum_n \frac{2v}{Lk_n\omega_n}\sin(k_n a)\sin\left(k_n\frac{\Delta\ell}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\sin(\omega_n t)$$

Suprimir el 7° armónico: $n=7 \Rightarrow B_7 = 0$

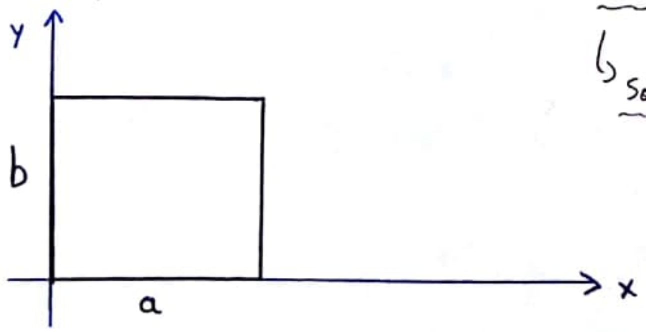
$$\sin(k_7 a) = 0 \Rightarrow k_7 a = \pi m \longrightarrow m = 1, 2, 3, \dots$$

$$\frac{7\pi}{L}a = \pi m$$

$$\boxed{a = \frac{mL}{7}} \rightarrow$$ hay que golpear la cuerda a $\frac{\ell}{7}, \frac{2\ell}{7}, \frac{3\ell}{7}$ del extremo

⑦ Modos normales de oscilación de una membrana elástica con un marco rígido de lados a y b?

¿Qué pasa si $a = b$?



Ec ondas 2D: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

↳ solución general:

$$u(x,y,t) = A\, e^{i(k_x x + k_y y - \omega t)} + B\, e^{i(-k_x x + k_y y - \omega t)} + C\, e^{i(k_x x - k_y y - \omega t)} + D\, e^{i(-k_x x - k_y y - \omega t)}$$

4 Ejs

Condiciones de contorno :

[1] $u(x=0, y, t) = 0 \quad \forall y, t$

[2] $u(x=a, y, t) = 0 \quad \forall y, t$

[3] $u(x, y=0, t) = 0 \quad \forall x, t$

[4] $u(x, y=b, t) = 0 \quad \forall x, t$

$u(x=0, y=0, t) = 0$

$\downarrow$

$u(x=0, y=0, t) = A e^{-i\omega t} + B e^{-i\omega t} + C e^{-i\omega t} + D e^{-i\omega t} = 0 \quad \forall t$

$B + D = -(A+C) \longleftarrow \boxed{A + B + C + D = 0} \longrightarrow (A+B) = -(C+D)$

[1] $u(0,y,t) = \left(A e^{ik_y y} + B e^{ik_y y} + C e^{-ik_y y} + D e^{-ik_y y}\right) e^{-i\omega t} = 0$

$\forall t \quad \hookrightarrow \quad (A+B) e^{ik_y y} + (C+D) e^{-ik_y y} = 0$

[2] $u(x,0,t) = \left(A e^{ik_x x} + B e^{-ik_x x} + C e^{ik_x x} + D e^{-ik_x x}\right) e^{-i\omega t} = 0$

$\forall t \quad \Rightarrow \quad (A+C) e^{ik_x x} + (B+D) e^{-ik_x x} = 0$

[1] $(A+B) e^{ik_y y} - (A+B) e^{-ik_y y} = 0 , \quad \underbrace{(A+B)\left[e^{ik_y y} - e^{-ik_y y}\right] = 0 \quad \forall y}$

$\Downarrow$

$A + B = 0 \Rightarrow \boxed{B = -A}$

[2] $\underline{(A+C) e^{ik_x x} - (A+C) e^{-ik_x x} = 0 \quad \forall x}$

$(A+C)\left[e^{ik_x x} - e^{-ik_x x}\right] = 0 \quad \forall x \Rightarrow (A+C) = 0 , \quad \boxed{C = -A}$

$A + B = -(D + C)$

$\boxed{D = -(A + B + C) = -(A - A - A) = A}$

$$\left\{ \begin{array}{ll} A & B = -A \\ C = -A & D = A \end{array} \right.$$

Solución: $y(x,t) = A\left[e^{i(k_x x + k_y y - \omega t)} - e^{i(-k_x x + k_y y - \omega t)} - \right.$

$\left. - e^{i(k_x x - k_y y - \omega t)} + e^{i(-k_x x - k_y y - \omega t)} \right] =$

$= A e^{-i\omega t}\left[\left(e^{ik_x x} - e^{-ik_x x}\right) e^{ik_y y} - \left(e^{ik_x x} - e^{-ik_x x}\right) e^{-ik_y y} \right] =$

$= A e^{-i\omega t} \left(e^{ik_x x} - e^{-ik_x x}\right)\left(e^{ik_y y} - e^{-ik_y y}\right) =$

$= A e^{-i\omega t}\, 2i \sin k_x x \; 2i \sin k_y y = \boxed{-4A\, e^{-i\omega t} \sin k_x x \sin k_y y}$

$u(x=a, y, t) = 0 \quad \Rightarrow \quad -4A\, e^{-i\omega t} \sin k_x a \; \sin k_y y = 0 \quad \forall y, t$

$\forall y, t$

$\sin k_x a = 0 \Rightarrow k_x a = n\pi$

$\boxed{k_{x_n} = \frac{n\pi}{a}}$

$n = 1, 2, \ldots$

modos normales

$u(x, y=b, t) = 0 \quad \Rightarrow \quad -4A\, e^{-i\omega t} \sin k_x x \; \sin k_y b = 0 \quad \forall x, t$

$\sin k_y b = 0 \; . \; k_y b = m\pi$

$\boxed{k_{y_m} = \frac{m\pi}{b}}$ $\quad m = 1, 2, \ldots$

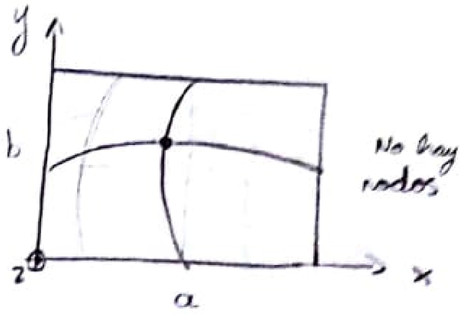
Parte real como solución:

$$\boxed{u(x,y,t) = -4A \sin \frac{n\pi}{a} x \, \sin \frac{n\pi}{b} y \, \cos \omega t}$$

Algunos modos:

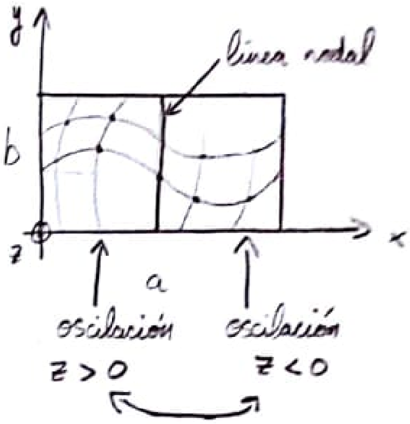
$\rightarrow$ $n = m = 1 \Rightarrow \begin{cases} k_x = \frac{\pi}{a} \Rightarrow \lambda_x = 2a \\ k_y = \frac{\pi}{b} \Rightarrow \lambda_y = 2b \end{cases}$

$$u(x,y,t) = -4A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \cos\omega t$$



$\rightarrow$ $n = 2,\ m = 1 \Rightarrow \begin{cases} k_x = \frac{2\pi}{a} \Rightarrow \lambda_x = a \\ k_y = \frac{\pi}{b} \Rightarrow \lambda_y = 2b \end{cases}$

$$u(x,y,t) = -4A \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} \cos\omega t$$



$\rightarrow$

$$k^2 = k_x^2 + k_y^2 = \frac{\omega^2}{v^2}$$

$$\omega = v\sqrt{k_x^2 + k_y^2} = v\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} = v\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$\boxed{\omega_{n,m} = v\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}} \rightarrow \text{frecuencias características}$$

$(n \times m) \rightarrow (\infty)$

⊛ Si $\boxed{a = b} \longrightarrow \boxed{\omega_{n,m} = \frac{v\pi}{a}\sqrt{n^2 + m^2}}$ $\qquad \omega_{n,m} = \omega_{m,n}$

¿Líneas nodales y oscilación para la combinación de dos oscilaciones de igual amplitud correspondientes a los modos normales 2 y 3 cuando se ordenan de frecuencia menor a mayor?

Frecuencias de los modos normales:

- $n = 1,\ m = 1 \rightarrow \omega_{11} = \frac{v\pi}{a}\sqrt{2}$ (fundamental) **Modo 1**

$\left.\begin{array}{l} \bullet\ n = 1,\ m = 2 \\ \bullet\ n = 2,\ m = 1 \end{array}\right\} \rightarrow \omega_{12} = \omega_{21} = \frac{v\pi}{a}\sqrt{5}$ modos degenerados (misma frecuencia) **Modos 2 y 3**

• $n=2,\ m=2 \implies \omega_{22} = \frac{v\pi}{a}\sqrt{8}$ Modo 4

Modo 2 : $n=1$ y $m=2 \implies u(x,y,t) = A \sin\frac{\pi}{a}x \sin\frac{2\pi}{a}y \cos\omega t$

Modo 3 : $n=2$ y $m=1 \implies u(x,y,t) = A \sin\frac{2\pi}{a}x \sin\frac{\pi}{a}y \cos\omega t$

Superposición :

$$u(x,y,t) = A\cos\omega t\left[\sin\frac{\pi}{a}x \sin\frac{2\pi}{a}y + \sin\frac{2\pi}{a}x \sin\frac{\pi}{a}y\right] =$$

$$= A\cos\omega t\left[\sin\frac{\pi}{a}x\ 2\sin\frac{\pi}{a}y\cos\frac{\pi}{a}y + 2\sin\frac{\pi}{a}x\cos\frac{\pi}{a}x\sin\frac{\pi}{a}y\right] =$$

$$= 2A\cos\omega t\ \sin\frac{\pi}{a}x\ \sin\frac{\pi}{a}y\left[\cos\frac{\pi}{a}x + \cos\frac{\pi}{a}y\right] =$$

(*) $\cos\alpha+\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos\alpha-\beta = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

$\alpha = \frac{(\alpha+\beta)+(\alpha-\beta)}{2}$ $\quad \beta = \frac{(\alpha+\beta)-(\alpha-\beta)}{2}$

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$= 2A\cos\omega t \sin\frac{\pi}{a}x\ \sin\frac{\pi}{a}y \cdot 2 \cdot \cos\frac{\pi}{a}\left(\frac{x+y}{2}\right)\cdot\cos\frac{\pi}{a}\left(\frac{x-y}{2}\right)$$

$$\boxed{u(x,y,t) = 4A\cos\omega t\ \sin\frac{\pi}{a}x\ \sin\frac{\pi}{a}y\ \cos\frac{\pi}{a}\left(\frac{x+y}{2}\right)\cos\frac{\pi}{a}\left(\frac{x-y}{2}\right)}$$

Líneas nodales $\longrightarrow$ $u(x,y,t) = 0\ \forall t$:

① $\sin\frac{\pi}{a}x = 0 \implies x = 0$
$x = a$

② $\sin\frac{\pi}{a}y = 0 \implies y=0$
$y=a$

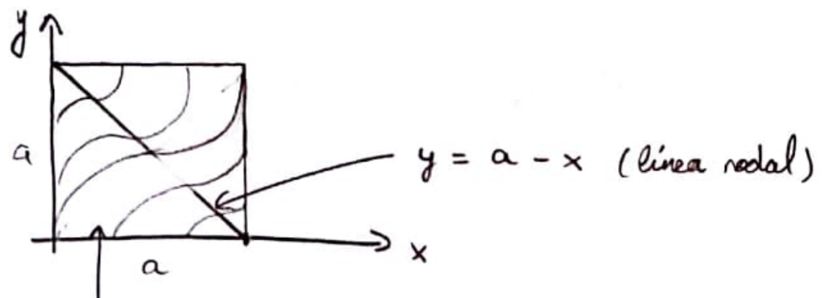
③ $\cos\frac{\pi}{a}\left(\frac{x+y}{2}\right) = 0 \implies \frac{\pi}{a}\left(\frac{x+y}{2}\right) = (2n+1)\frac{\pi}{2}$
$\frac{x+y}{a} = 2n+1$
$x+y = (2n+1)a$

$n=0 \qquad x+y=a\,,\quad \boxed{y=a-x} \rightarrow$ Línea nodal

$n=1 \qquad x+y=3a \quad \otimes$

(4) $\cos\frac{\pi}{2}\left(\frac{x-y}{a}\right)=0 \Rightarrow x-y=a(2n+1)$

$n=0 \qquad x-y=a\,,\quad x=y+a \quad \otimes$



Cuando este sector triangular oscila hacia arriba el otro oscila hacia abajo

(8) Olas del mar $\longrightarrow$ $\boxed{\text{dispersión}}$ cuando la profundidad es grande comparada con la longitud de onda.

su forma se altera durante la propagación

$g = 10\,ms^{-2}$

Relación de dispersión : $\boxed{\omega^2 = gk} \rightarrow$ No lineal (dispersión)

Barco a motor $\longrightarrow$ avanza hacia la playa

Grupo de olas originado en la proa $\longrightarrow$ se propaga hacia la playa con $v_g = 10\,m/s$

¿ Velocidad (v_f) de una ola con $\lambda = 1\,m$?

$$\omega = \sqrt{gk}$$

$$v_f(k) = \frac{\omega}{k} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}}$$

$$v_g(k) = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{gk} = \frac{1}{2}\frac{g}{\sqrt{gk}} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

dispersión normal

$$\boxed{v_f(k) > v_g(k) \quad \forall k}$$

$$\boxed{v_f(k) = \sqrt{\frac{g}{k}}} \quad , \quad v_f(\lambda) = \sqrt{\frac{g\lambda}{2\pi}} \quad , \quad v_f(\lambda = 1m) = \sqrt{\frac{10\, m/s^2 \cdot 1m}{2\pi}} \simeq \boxed{1'25\ m/s}$$

$$v_g = \frac{1}{2}\sqrt{\frac{g}{k_0}} \quad , \quad k_0 = \left(\frac{\sqrt{g}}{2v_g}\right)^2 = \frac{g}{4v_g^2} \quad , \quad \boxed{k_0 = \frac{10\,(m/s^2)}{4\cdot(10\, m/s)^2} = 0'025\ m^{-1}}$$

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0} \longrightarrow$$ Centro del intervalo en el que $\underbrace{A(k)}$ toma valores no nulos

amplitudes de las ondas que contribuyen al paquete

$$\boxed{\lambda_0 = \frac{2\pi}{k_0} \simeq 250\ m}$$

(9) Dispersión en aguas poco profundas : $\boxed{\omega^2 = gk + \frac{\tau}{\rho}k^3}$

$g \simeq 10\ ms^{-2}$

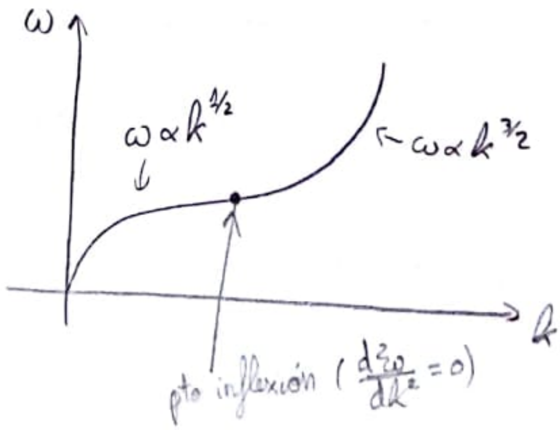
$\rho \simeq 1\ g/cm^3 \quad \frac{10^{-3}Kg}{1g}\cdot\frac{1cm^3}{10^{-6}m^3} = 1000\,Kg/m^3$

$\tau \simeq 0'074\ N/m$

$\tau \equiv$ tensión superficial

a) Representar gráficamente ω frente a k

$$\omega = \sqrt{gk + \frac{\tau}{\rho}k^3}$$



b) ¿Para qué longitudes de onda es importante la tensión superficial?

$$g k < \frac{\tau}{\rho} k^3 \ , \quad g < \frac{\tau}{\rho} k^2 , \quad k^2 > \frac{g\rho}{\tau} \ , \quad \frac{4\pi^2}{\lambda^2} > \frac{g\rho}{\tau} \ .$$

$$\lambda^2 < \frac{4\pi^2 \tau}{g\rho} \ , \quad \lambda < 2\pi\sqrt{\frac{\tau}{g\rho}}$$

$$2\pi\sqrt{\frac{0'074 \ N/m}{10 \ m/s^2 \cdot 1000 \ Kg/m^3}} \simeq 0'0171 \ m \longrightarrow \boxed{\lambda < 1'71 \ cm}$$

La tensión superficial influye en la propagación

c) ¿Es un medio dispersivo?

Sí, porque la relación $\omega(k)$ no es lineal.

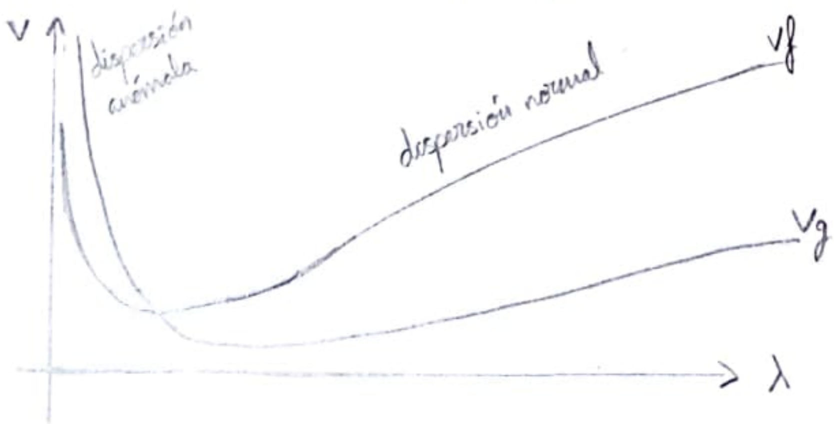
Podemos calcular:

$$v_f = \frac{\omega}{k} = \frac{\sqrt{gk + \tau/\rho \, k^3}}{k} = \sqrt{\frac{g}{k} + \frac{\tau}{\rho} k}$$

$$\omega(k) = \sqrt{gk + \frac{\tau}{\rho} k^3}$$

$$v_g = \frac{d\omega}{dk} = \frac{g + 3\frac{\tau}{\rho} k^2}{2\sqrt{gk + \tau/\rho \, k^3}}$$

$\boxed{v_g \neq v_f}$
$\Downarrow$
medio dispersivo



d) Longitudes de onda para las que v_f y v_g son mínimas:

$$v_f = \sqrt{\frac{g}{k} + \frac{Z}{\rho} k}$$

$$\left.\frac{dv_f}{dk}\right|_{k_0} = 0 \; ? \qquad \frac{dv_f}{dk} = \frac{-\frac{g}{k^2} + \frac{Z}{\rho}}{2\sqrt{\frac{g}{k} + \frac{Z}{\rho} k}}$$

$$\left.\frac{dv_f}{dk}\right|_{k_0} = \frac{-g/k_0^2 + \frac{Z}{\rho}}{2\sqrt{g/k_0 + \frac{Z}{\rho} k_0}} = 0 \Rightarrow \frac{-g}{k_0^2} + \frac{Z}{\rho} = 0, \quad \frac{g}{k_0^2} = \frac{Z}{\rho}$$

$$k_0^2 = \frac{g\rho}{Z}$$

$$\lambda_0 = 2\pi\sqrt{\frac{Z}{g\rho}} \Leftarrow k_0 = \sqrt{\frac{g\rho}{Z}} = \frac{2\pi}{\lambda_0}$$

$$\boxed{\lambda_0 \simeq 1'71 \text{ cm}}$$

→ longitud de onda de la onda que viaja con menor velocidad.

$$v_g = \frac{g + \frac{3Z}{\rho}k^2}{2\sqrt{gk + \frac{Z}{\rho}k^3}}$$

$$\left.\frac{dv_g}{dk}\right|_{k_0} = \left.\frac{d^2\omega}{dk^2}\right|_{k_0} = \left.\frac{\frac{6Z}{\rho}k\, 2\sqrt{gk + \frac{Z}{\rho}k^3} - \left(g + \frac{3Z}{\rho}k^2\right)\frac{\left(g + \frac{3Z}{\rho}k^2\right)}{\sqrt{gk + \frac{Z}{\rho}k^3}}}{2\left(gk + \frac{Z}{\rho}k^3\right)}\right|_{k_0} =$$

punto de inflexión de $\omega(k)$

$$= \left.\frac{\frac{12Z}{\rho}k\left(gk + \frac{Z}{\rho}k^3\right) - \left(g + \frac{3Z}{\rho}k^2\right)^2}{2\left(gk + \frac{Z}{\rho}k^3\right)\sqrt{gk + \frac{Z}{\rho}k^3}}\right|_{k_0} = 0$$

$$\frac{12Z}{\rho}k_0\left(gk_0 + \frac{Z}{\rho}k_0^3\right) - \left(g + \frac{3Z}{\rho}k_0^2\right)^2 = 0$$

$$12\left(\frac{Z}{\rho}k_0^2\right)\left(g + \left(\frac{Z}{\rho}k_0^2\right)\right) - \left(g + 3\left(\frac{Z}{\rho}k_0^2\right)\right)^2 = 0$$

Cambio $x = \frac{Z}{e} k_0^2$:

$$12x(g+x) - (g+3x)^2 = 0$$

$$12gx + 12x^2 - g^2 - 9x^2 - 6gx = 0$$

$$3x^2 + 6gx - g^2 = 0 \; , \quad x = \frac{-6g \pm \sqrt{36g^2 + 12g^2}}{6} =$$

$$= -g \pm \frac{\sqrt{48}}{6} g = \left(-1 \pm \frac{\sqrt{48}}{6}\right) g$$

$$\boxed{x > 0}$$

$$x = \frac{Z}{e} k_0^2 \quad x = \left(-1 + \frac{\sqrt{2^4 \cdot 3}}{6}\right) g = \left(-1 + \frac{2}{3}\sqrt{3}\right) g$$

$$k_0^2 = \frac{eg}{Z}\left(\frac{2}{3}\sqrt{3} - 1\right)$$

$$\underbrace{k_0 = \sqrt{\frac{ge}{Z}}\sqrt{\left(\frac{2}{3}\sqrt{3} - 1\right)} = \frac{2\pi}{\lambda_0}}$$

k_0 en torno al que las amplitudes de las ondas que contribuyen al paquete no son nulas.

$$\lambda_0 = 2\pi \sqrt{\frac{Z}{ge}} \frac{1}{\left(\frac{2}{3}\sqrt{3} - 1\right)^{1/2}} \simeq 0'043 \text{ m}$$